Basic Comparison Test

The idea of the basic comparison test is to learn something about a series by comparing it to another series, about which we know whether it converges or diverges.

Basic Comparison Test: Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences such that

$$(*) \qquad 0 \leq a_n \leq b_n, \quad \text{for all } n \geq N.$$

Then

(1) If the series $\displaystyle\sum_{n=0}^{\infty} b_n$ converges, then so does the series $\displaystyle\sum_{n=0}^{\infty} a_n$.

(2) If the series $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges to ∞, then so does the series $\displaystyle\sum_{n=0}^{\infty} b_n$.

Remark 1.1: This test implies that if the underlying sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of two given series satisfy the condition $(*)$, then a certain similarity in the behaviour of the two series can be observed.

In the first case we suppose that the series $\sum b_n$ converges, which means that the sum of all b_n is finite. So intuitively it makes sense that the sum over the smaller elements a_n (recall that $(*)$ says that in general $a_n \leq b_n$) must also be finite. In particular, $\sum a_n$ also converges.

On the other hand if the series $\sum a_n$ diverges to infinity, (that is, the sum of all a_n is infinity), then surely the sum of the larger elements b_n must also be infinity. In particular, $\sum b_n$ also diverges to infinity.

What series to take for the comparison is not always an easy decision and may require several tries. Also at the start it is not clear which of the two cases of the basic comparison test will work (if at all). As a general indicator one should aim for the two underlying sequences (a_n) and (b_n) to be very similar.

Material developed by the Department of Mathematics & Statistics, NUIM and supported by www.ndlr.com.

Example 1.2: Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$. For large n the sequence $\left(\frac{1}{2^n+1}\right)_{n\geq 1}$ behaves similarly to the sequence $\left(\frac{1}{2^n}\right)_{n\geq 1}$. (By that we mean that as n gets bigger, the difference between dividing 1 by 2^n+1 and dividing 1 by 2^n tends towards zero. For instance there is a noticeable difference in dividing 1 by two or three, but much less so if we divide 1 by a million or a million and one. That means the larger the value we are dividing by, the less it matters if we increase this value by 1.)

Furthermore we have that

$$0 \leq \frac{1}{2^n+1} \leq \frac{1}{2^n}, \quad \text{for all } n \geq 1, \text{ and the series } \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges,}$$

since it is a geometric series (see Example 2.3 on the handout Geometric Series II). Thus the first part of the basic comparison test implies that

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \qquad \text{also converges.}$$

Example 1.3: Consider the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$. For large n the sequence $\left(\frac{1}{\sqrt{n}-1}\right)_{n\geq 2}$ behaves like the sequence $\left(\frac{1}{\sqrt{n}}\right)_{n\geq 2}$. Also we have

$$0 \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n}-1}, \quad \text{for all } n \geq 2.$$

Furthermore observe that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$. In particular this series diverges to infinity, and thus so does the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$. (For more details refer to the handouts on p-series.) Now it follows from the second part of the basic comparison test that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1} \qquad \text{also diverges to infinity.}$$