BASIC COMPARISON TEST II (OF III)

Remark 2.1: It is important to realize that the basic comparison test cannot be reversed. For instance if the series $\sum b_n$ diverges to infinity, then we cannot draw any conclusions about the series $\sum a_n$. Vice versa if $\sum a_n$ converges, then this test says nothing about $\sum b_n$.

Example 2.2: In the following let $n \geq 1$. We set $b_n = 1$. Then the series $\sum_{n=1}^{\infty} 1$ is a geometric series for $x = 1$, and thus diverges to infinity, (see Example 2.1 on the handout Geometric Series II).
Next let $a_n = \frac{1}{n}$. Then $a_n \leq b_n$ and since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, it diverges too, (for more information about the harmonic series see Example 1.1. on the handout p-Series I and Example 2.1 on the handout p-Series II.).
Finally let $a'_n = \frac{1}{n^2}$. Again we have $a'_n \leq b_n$, but the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series for $p = 2$, and thus it converges, (see Example 2.2. on the handout p-Series II).
In particular the divergence of $\sum b_n$ has no influence on the behavior of $\sum a_n$ or $\sum a'_n$, and neither those the basic comparison test claim that such an influence exists.

Example 2.3: Consider the series $\sum_{n=0}^{\infty} \frac{1}{2n+1}$. For large n the sequence $a_n = \frac{1}{2n+1}$ behaves similarly to the sequence $b_n = \frac{1}{n}$. Also we observe that $0 \leq a_n \leq b_n$, for all $n \geq 1$. As $\sum b_n$ is the harmonic series, we know it diverges. Hence in this situation the basic comparison test cannot be applied.

That means we have to try to find another series for the comparison. Note that $\left(\frac{1}{2n+1}\right)_{n\geq 1}$ also behaves similarly to $\left(\frac{1}{2(n+1)}\right)_{n\geq 1}$. Here

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we have

$$0 \leq \frac{1}{2(n+1)} \leq \frac{1}{2n+1}, \quad \text{for all } n \geq 0, \text{ and}$$

$$\sum_{n=0}^{\infty} \frac{1}{2(n+1)} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{n+1} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

That is, $\sum_{n=0}^{\infty} \frac{1}{2(n+1)}$ diverges to infinity, and so the second part of the basic comparison test implies that $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ also diverges to infinity.

Remark 2.4: Observe that in the basic comparison test we only need to show that $0 \leq a_n \leq b_n$ is true for all $n \geq N$, where N is some finite integer. That means a finite number of initial elements of both sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ may behave as they want, as long as they satisfy $0 \leq a_n \leq b_n$ from some element onwards (here these elements are a_N and b_N). This is due to the fact that the beginning of a sequence has no influence on the convergence or divergence of its corresponding series.

Example 2.5: Consider the series $\sum_{n=1}^{\infty} \frac{\ln \sqrt{n}}{n}$. First we observe that $\ln \sqrt{n} \geq 1$, for all $n \geq e^2 \approx 7.389$. Hence we could try to compare the sequence $\left(\frac{\ln \sqrt{n}}{n}\right)_{n\geq 1}$ to the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$. In fact we have

$$0 \leq \frac{1}{n} \leq \frac{\ln \sqrt{n}}{n}, \quad \text{for all } n \geq 8.$$

Note that the inequality only holds for all $n \geq 8$, that is, it does not hold for the first seven sequence elements. However, as we discussed in Remark 2.4, for the convergence or divergence of a series it does not matter what happens in the beginning of the underlying sequence.

Furthermore recall that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity. Thus the second part of the basic comparison test implies that $\sum_{n=1}^{\infty} \frac{\ln \sqrt{n}}{n}$ also diverges to infinity.