ALTERNATING SERIES I (OF III)

Definition of an alternating series

A series $\sum_{n=0}^{\infty} a_n$ is called an **alternating series** if in the underlying sequence $(a_n)_{n\geq 0}$ any two consecutive elements have a different sign.

Example 1.1: Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Here the underlying sequence is given by $a_n = \frac{1}{n}$, for $n \geq 1$. Hence the elements in this sequence are

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad a_5 = \frac{1}{5}, \quad a_6 = \frac{1}{6}, \quad \ldots$$

As all these elements have the same sign, (in this case, they are all positive) the harmonic series is <u>not</u> an alternating series.

Example 1.2: Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Here the underlying sequence is given by $a_n = \frac{(-1)^n}{n}$, for $n \geq 1$. Note that if n is odd we have $(-1)^n = -1$, whereas, whenever n is even we get $(-1)^n = 1$. Hence the elements in our sequence are

$$a_1 = -1, \quad a_2 = \frac{1}{2}, \quad a_3 = -\frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad a_5 = -\frac{1}{5}, \quad a_6 = \frac{1}{6}, \quad \ldots$$

Thus the first, third, fifth (and so on) elements of the sequence are negative while the second, fourth, sixth (and so on) elements of the sequence are positive. In particular any two consecutive elements, say $a_3 = -\frac{1}{3}$ and $a_4 = \frac{1}{4}$, or $a_{100} = \frac{1}{100}$ and $a_{101} = -\frac{1}{101}$, have different signs. Therefore the given series is alternating.

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Example 1.3: Consider the series

$$\sum_{n=0}^{\infty} \cos(n\pi).$$

If we study the term $\cos(n\pi)$ we see that

$$\cos(0\cdot\pi)=1,\quad \cos(1\cdot\pi)=-1,\quad \cos(2\cdot\pi)=1,\quad \cos(3\cdot\pi)=-1,$$
$$\cos(4\cdot\pi)=1,\quad \cos(5\cdot\pi)=-1,\quad \ldots$$

In particular any two consecutive terms have different signs. Therefore the given series is alternating.

Example 1.4: In the previous example we have seen that any two consecutive terms in the sequence $(\cos(n\pi))_{n\geq 0}$ have a different sign. Now suppose that $(a_n)_{n\geq 0}$ is a sequence of non-zero elements that all have the same sign, that is, either $a_n > 0$, for all $n \geq 0$ or $a_n < 0$, for all $n \geq 0$. Since all a_n have the same sign any two consecutive terms in the sequence $(a_n \cos(n\pi))_{n\geq 0}$ still must have a different sign. Therefore the series

$$\sum_{n=0}^{\infty} a_n \cos(n\pi)$$

is alternating.

For instance we may chose $a_n = 5$, for all $n \geq 0$. Then $a_n > 0$, for all $n \geq 0$, and thus

$$\sum_{n=0}^{\infty} 5\cos(n\pi)$$

is alternating.

If we let $a_n = 2^n$, for all $n \geq 0$, then $a_n > 0$, for all $n \geq 0$, and thus

$$\sum_{n=0}^{\infty} 2^n \cos(n\pi)$$

is alternating.

Also we can let $a_n = \frac{-1}{n+1}$, for all $n \geq 0$. Then $a_n < 0$, for all $n \geq 0$, and thus

$$\sum_{n=0}^{\infty} \frac{-1}{n+1}\ \cos(n\pi)$$

is alternating.