ALTERNATING SERIES II (OF III)

How to recognize an alternating series?

By definition a series is alternating if any two consecutive elements of the underlying sequence have different signs. As this condition has to be satisfied by the first and the second element of the underlying sequence as well as by the one-millionth and one-million and first element it is not always so easy to recognize an alternating series.

In the following we look at a family of alternating series that is easily recognisable. Assume we have a sequence $(b_n)_{n\geq 0}$ of non-zero elements which all have the same sign. That means $b_n > 0$, for all $n \geq 0$ or $b_n < 0$, for all $n \geq 0$. Then the corresponding series $\sum_{n=0}^{\infty} b_n$ is not alternating because there is no change of sign at all. Next consider the series

$$\sum_{n=0}^{\infty} (-1)^n \; b_n.$$

Then the elements of the underlying sequence of this series are

$$(-1)^0 \; b_0 = b_0, \quad (-1)^1 \; b_1 = -b_1, \quad (-1)^2 \; b_2 = b_2, \quad (-1)^3 \; b_3 = -b_3,$$
$$(-1)^4 \; b_4 = b_4, \quad (-1)^5 \; b_5 = -b_5, \quad \ldots$$

Observe how the sign changes from element to element (recall that all b_n have the same sign). This change of sign is a consequence of the term $(-1)^n$, which changes the sign every time we increase n by one.

The same effect occurs if we consider the series

$$\sum_{n=0}^{\infty} (-1)^{n+1} \; b_n.$$

Then the elements of the underlying sequence of this series are

$$-b_0, \quad b_1, \quad -b_2, \quad b_3, \quad -b_4, \quad b_5, \quad \ldots$$

Again the term $(-1)^{n+1}$ is responsible for the change of sign, but this only works because the term b_n does not change its sign.

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Example 2.1: Consider the series

$$\sum_{n=0}^{\infty}(-1)^{n+1}\ \frac{n}{n+1}.$$

As the term $\frac{n}{n+1}$ is always positive and the term $(-1)^{n+1}$ constantly changes its sign, we can conclude that our series is alternating.

Example 2.2: Consider the series

$$\sum_{n=0}^{\infty}(-2)^n\ n!.$$

Here the underlying sequence is given by $b_n = (-2)^n\ n!$, for $n \geq 0$. We can rewrite b_n as

$$b_n = (-2)^n\ n! = (-1)^n\ 2^n\ n!.$$

Hence we have the term $2^n\ n!$, which is positive, for all $n \geq 0$, and we have the term $(-1)^n$ in charge of changing the sign. Therefore the series $\sum_{n=0}^{\infty}(-2)^n\ n!$ is an alternating series. (Note that the two series $\sum_{n=0}^{\infty}(-2)^n\ n!$ and $\sum_{n=0}^{\infty}(-1)^n\ 2^n\ n!$ are equivalent)

Example 2.3: Consider the series

$$\sum_{n=-10}^{\infty}\left(-\frac{1}{100}\right)^n.$$

Here the underlying sequence is given by $b_n = \left(-\frac{1}{100}\right)^n$, for $n \geq -10$. Again we can rewrite b_n as

$$b_n = \left(-\frac{1}{100}\right)^n = (-1)^n\ \frac{1}{100^n}.$$

The term $\frac{1}{100^n}$ is positive, for all $n \geq -10$, while the term $(-1)^n$ keeps changing sign. Therefore the given series is an alternating series.

Example 2.4: Note that a series of the form $\sum(-1)^n\ b_n$ is only alternating if all b_n have the same sign. For instance consider the series

$$\sum_{n=0}^{\infty}(-1)^n\ \cos(n\pi).$$

Here the term $\cos(n\pi)$ alternates between plus and minus one. In fact we have that $(-1)^n \cdot \cos(n\pi) = 1$, for all $n \geq 0$, and thus the series is not alternating.