Convergence of an alternating series

The following test describes conditions under which an alternating series converges. This test is known as the **Alternating Series Test**.

An alternating series $\sum_{n=0}^{\infty} a_n$ converges if the following two conditions are satisfied:

(1) $|a_n| \geq |a_{n+1}|$, for all $n \geq N$, where N is some integer.

(2) $\lim_{n\to\infty} |a_n| = 0$

Remark 3.1: Note that property (1) means that the sequence $(|a_n|)_{n\geq 0}$ is decreasing (with the possible exception of the first N elements).

Example 3.2: Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. In Example 1.2 we have verified that this series is alternating. Furthermore the underlying sequence is given by $a_n = \frac{(-1)^n}{n}$, for $n \geq 1$. The alternating series test requires us to work with the absolute value of a_n. In our case that means $|a_n| = \frac{1}{n}$. We check the first condition in the test and obtain

$$|a_n| = \frac{1}{n} > \frac{1}{n+1} = |a_{n+1}|.$$

The inequality in the middle is a consequence of $n + 1 > n$, for all $n \geq 1$. In particular we have shown that $|a_n| \geq |a_{n+1}|$, for all $n \geq 1$.

Next we take a look at the second condition. Here we study the limit of the sequence $(|a_n|)_{n\geq 0}$. In our case we get

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore both conditions in the alternating series test are satisfied. In particular we conclude that our given alternating series converges.

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Example 3.3: Consider the series $\sum_{n=-10}^{\infty}\left(-\frac{1}{100}\right)^n$. This series is alternating (see Example 2.3) and the underlying sequence is given by $a_n = \left(-\frac{1}{100}\right)^n$. Thus $|a_n| = \frac{1}{100^n}$, for $n \geq -10$. Since

$$|a_n| = \frac{1}{100^n} > \frac{1}{100^{n+1}} = |a_{n+1}|,$$

and

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{100^n} = 0,$$

both conditions of the alternating series test are satisfied. Consequently the series converges.

Remark 3.4: For a sequence $(a_n)_{n\geq 0}$ it holds that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$. That means if the second condition of the alternating series test is violated, that is, $\lim_{n\to\infty} |a_n| \neq 0$, then we immediately know that $\lim_{n\to\infty} a_n \neq 0$. In this case it follows from the divergence test that the series does not converge. (For more information refer to the handout on the divergence test.)

Example 3.5: The series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n}{n+1}$ is alternating (see Example 2.1.). Here the underlying sequence is $a_n = (-1)^{n+1}\frac{n}{n+1}$, and thus $|a_n| = \frac{n}{n+1}$, for all $n \geq 0$. Then

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0.$$

Hence the second condition in the alternating series test is violated and the alternating series test fails. However, as we discussed in Remark 3.4, we can now apply the divergence test to conclude that the series does not converge. In fact, more analysis shows that this series does not have a limit at all.

Example 3.6: The series $\sum_{n=0}^{\infty}(-2)^n\, n!$ is alternating (compare Example 2.2), and the underlying sequence is given by $a_n = (-2)^n\, n!$. Then $|a_n| = 2^n\, n!$, for $n \geq 0$. One can check that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} 2^n\, n! = \infty \neq 0.$$

Again the second condition of the alternating series test is violated, and thus the alternating series test fails. Now the divergence test can be applied to show that the series does not converge.