

Squeeze Theorem for Sequences

We discussed in the handout “Introduction to Convergence and Divergence for Sequences” what it means for a sequence to converge or diverge. We said that in order to determine whether a sequence $\{a_n\}$ converges or diverges, we need to examine its behaviour as n gets bigger and bigger. We also said the way we do this is to calculate $\lim_{n \rightarrow \infty} a_n$. Sometimes that limit can be difficult to calculate and we need to employ some other techniques. One of those techniques is to use the Squeeze Theorem for sequences. We begin with the statement of the theorem.

Squeeze Theorem for Sequences

If $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = L$ and there exists an integer N such that $b_n \leq a_n \leq c_n$ for all $n > N$, then $\lim_{n \rightarrow \infty} a_n = L$.

Example 1

In this example we want to determine if the sequence

$$\{a_n\} = \left\{ \frac{\sin(n)}{n} \right\}$$

converges or diverges.

First of all, recall that $-1 \leq \sin n \leq 1$ for all n . Therefore

$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \quad \text{as } n > 0 \text{ for all } n.$$

We choose $\{b_n\} = \left\{ -\frac{1}{n} \right\}$ and $\{c_n\} = \left\{ \frac{1}{n} \right\}$. We now have for this choice of $\{b_n\}$ and $\{c_n\}$, that $b_n \leq a_n \leq c_n$ for all n . Notice that $\lim_{n \rightarrow \infty} b_n = 0 = \lim_{n \rightarrow \infty} c_n$. Therefore by the Squeeze Theorem we can say that $\lim_{n \rightarrow \infty} a_n = 0$ also. In other words, the sequence $\{a_n\}$ converges to 0.

As a direct result of the Squeeze Theorem, we also have the Absolute Value Theorem.

Absolute Value Theorem

For the sequence $\{a_n\}$, if $\lim_{n \rightarrow \infty} |a_n| = 0$ then $\lim_{n \rightarrow \infty} a_n = 0$

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Example 2

Suppose we want to determine whether the sequence

$$\{b_n\} = \left\{ \frac{(-1)^n}{n^2 + 2} \right\}$$

converges or diverges. Using the Absolute Value Theorem we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n^2 + 2} \right| &= \lim_{n \rightarrow \infty} \frac{|(-1)^n|}{|n^2 + 2|} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2 + 2} \\ &= 0 \end{aligned}$$

Therefore by the Absolute Value Theorem, $\{b_n\} = \left\{ \frac{(-1)^n}{n^2+2} \right\}$ converges to 0.

Try the following exercises for practice. In each case, use the Squeeze Theorem or the Absolute Value Theorem to determine if the sequence converges or diverges.

(a)

$$\{a_n\} = \left\{ (-1)^n \frac{1}{n} \right\}$$

(b)

$$\{a_n\} = \left\{ 2 + \frac{\sin(n)}{n} \right\}$$

(c)

$$\{a_n\} = \left\{ 4 + \frac{\cos(n)}{\sqrt{n}} \right\}$$

(Hint: For parts (b) and (c), recall that $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all x .)

Solutions

(a) Converges to 0.

(b) Converges to 2.

(c) Converges to 4.