Using L'Hôpital's Rule to Find Limits of Sequences

We discussed in the handout "Introduction to Convergence and Divergence" what it means for a sequence to converge or diverge. We said that in order to determine whether a sequence converges or diverges, we need to examine its behaviour as n gets bigger and bigger. We also said the way to do this is to calculate the $\lim_{n\to\infty} a_n$. Sometimes the limit can be difficult to calculate and we need to try some other techniques. Consider the following example.

Example 1
Suppose we wish to determine whether the sequence

$$\{a_n\} = \left\{\frac{\sqrt{n}}{e^n}\right\}$$

converges or diverges. As usual, we want to examine the behaviour of this sequence as n gets bigger and bigger, in order to determine if it converges or diverges. We begin by noticing that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{e^n} = \frac{\lim_{n\to\infty}\sqrt{n}}{\lim_{n\to\infty} e^n}$$

will produce the indeterminate form $\frac{\infty}{\infty}$. The question is, how do we cope with this. Recall in the handout "Determining Convergence and Divergence of Sequences Using Limits", we said that if f is a function of a real variable such that $\lim_{x\to\infty} f(x) = L$ and if $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every possible integer n, then $\lim_{n\to\infty} a_n = L$. We use this along with a theorem called L'Hôpital's Rule which deals with our situation. We state a version of it here:

L'Hôpital's Rule

If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ produces an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$, provided the limit on the right of the equals sign exists and $g'(x) \neq 0$.

Note that you can use L'Hôpital's rule repeatedly in any one question. (See Example 2 below.)

In Example 1 above, we could let $f(x) = \sqrt{x}$ and $g(x) = e^x$. So $\frac{f(n)}{g(n)} = \frac{\sqrt{n}}{e^n} = a_n$. We can now write:

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \\ &= \lim_{x\to\infty} \frac{\frac{d}{dx}(\sqrt{x})}{\frac{d}{dx}(e^x)}\end{aligned}$$

Material developed by the Department of Mathematics & Statistics, N.U.I. Maynooth and supported by the NDLR (www.ndlr.com).

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\begin{aligned}
&= \lim_{x\to\infty} \frac{\frac{d}{dx}(x^{\frac{1}{2}})}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x} \\
&= \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x} = 0.
\end{aligned}
$$

Therefore we conclude that the sequence $\{a_n\}$ converges to 0.

Example 2

Once again, let's suppose we wish to determine whether the sequence

$$\{b_n\} = \left\{ \frac{(\ln n)^2}{n} \right\}$$

converges or diverges. We begin by remarking that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \frac{\lim_{n\to\infty}(\ln n)^2}{\lim_{n\to\infty} n}$$

will produce the indeterminate form $\frac{\infty}{\infty}$. Proceeding as in Example 1, we let $f(x) = (\ln x)^2$ and $g(x) = x$. So $\frac{f(n)}{g(n)} = \frac{(\ln n)^2}{n} = b_n$. We apply L'Hôpital's rule:

$$
\begin{aligned}
\lim_{n\to\infty} b_n = \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\frac{d}{dx}((\ln x)^2)}{\frac{d}{dx}(x)} \\
&= \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{1} \\
&= \lim_{x\to\infty} \frac{2\ln x}{x}.
\end{aligned}
$$

Notice that once again

$$\lim_{x\to\infty} \frac{2\ln x}{x} = \frac{\lim_{x\to\infty}(2\ln x)}{\lim_{x\to\infty} x}$$

will produce the indeterminate form $\frac{\infty}{\infty}$. We therefore use L'Hôpitals rule again.

$$
\begin{aligned}
\lim_{x\to\infty} \frac{\frac{d}{dx}(2\ln x)}{\frac{d}{dx}(x)} &= \lim_{x\to\infty} \frac{\frac{2}{x}}{1} \\
&= \lim_{x\to\infty} \frac{2}{x} \\
&= 0.
\end{aligned}
$$

Therefore we conclude that the sequence $\{b_n\}$ converges to 0.

Exercises

In each of the following, use L'Hôpital's Rule to determine if the sequence converges or diverges.

(a) $\{a_n\} = \left\{ \frac{\ln(n+1)}{\sqrt{n}} \right\}$ (b) $\{a_n\} = \left\{ \frac{n}{\ln(n)} \right\}$ (c) $\{a_n\} = \left\{ \frac{e^{2n}}{n} \right\}$

(d) $\{a_n\} = \left\{ \frac{x^2}{e^{5x}} \right\}$

Solutions

(a) Converges to 0. (b) Sequence diverges. (c) Sequence diverges.

(d) Converges to 0.