Introduction to Convergence and Divergence for Sequences

We discussed in the handout "Introduction to Sequences" that we are often concerned about patterns in sequences. When dealing with infinite sequences, a reasonable question to ask would be, if one was to continue down the list of numbers, would we pass a point where the terms of the sequence begin to stabilise and get closer and closer to some finite number L, where $L \in \mathbb{R}$. If this happens we say that the sequence converges to L. If this does not happen, we simply say that the sequence diverges.

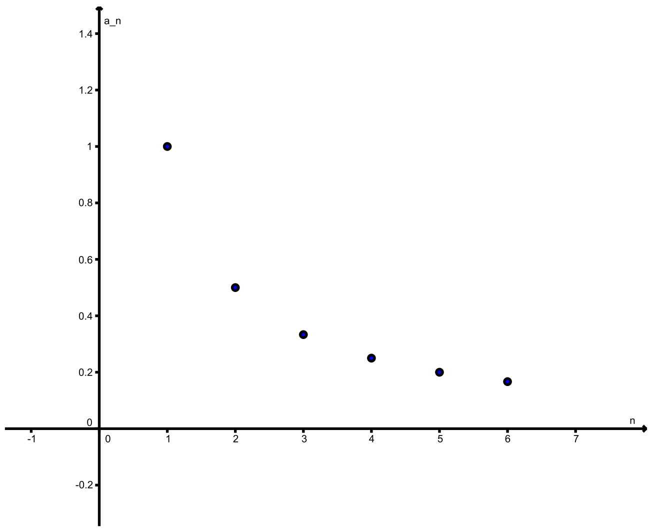
Consider the sequence

$$\{a_n\} = \left\{\frac{1}{n}\right\}.$$

We begin by writing out the first few terms of the sequence:

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...\right\}.$$

Examining the list above, the terms of the sequence seem to be getting smaller and smaller and appear to be tending towards 0. We might speculate that the sequence $\{a_n\}$ converges to 0. Below is a plot of the first six elements of the sequence $\{a_n\}$. We see that the graph agrees with what we said above. Visually we can see that the sequence seems to be converging towards 0. That is, as n gets bigger, a_n gets closer to 0.



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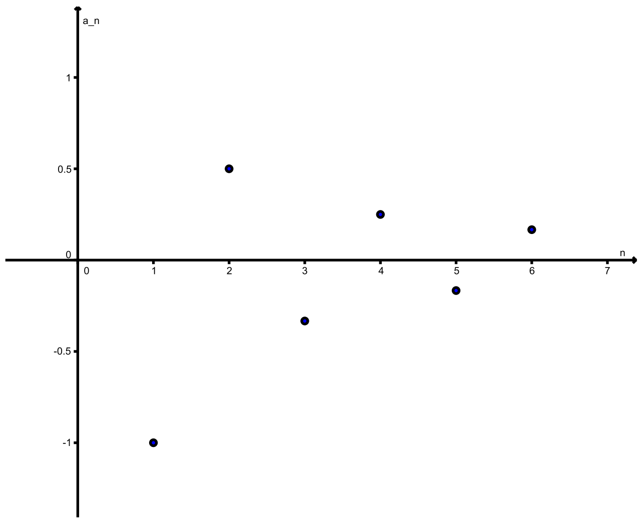
Next, consider the sequence

$$\{b_n\} = \left\{ \frac{(-1)^n}{n} \right\}.$$

We begin by writing out the first few terms of the sequence:

$$\{b_n\} = \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, ... \right\}.$$

Studying the list above, it is not immediately obvious if the sequence is converging or diverging. Below is a plot of the first 6 terms in the sequence.



Examining the graph, we see that early on in the sequence, the numbers are behaving quite erratically but as we move down along the list, they seem to be stabilising and converging to 0. The lesson here is that how a sequence behaves early on in the list does not really matter in determining if the sequence converges or diverges.

In general we need to examine the behaviour of the sequence as n gets bigger and bigger, in order to determine if it converges or diverges. What we have done above is not sufficient to prove either $\{a_n\}$ or $\{b_n\}$ converges to 0 but we will see in later handouts that the way we examine the behaviour of a sequence $\{a_n\}$, as n gets bigger and bigger, is to take the $\lim_{n\to\infty} a_n$.

In the following exercises, write out the first 10 terms of the sequence and from that, decide if you suspect the sequence converges or diverges.

(a) $\{a_n\} = \{2n\}$ (b) $\{a_n\} = \{2^n\}$ (c) $\{a_n\} = \left\{(-1)^n \frac{2}{n}\right\}$
(d) $\{a_n\} = \{(-2)^n\}$ (e) $\{a_n\} = \left\{\frac{2}{n^2}\right\}$