Determining Convergence and Divergence of Sequences Using Limits

We discussed in the handout "Introduction to Convergence and Divergence for Sequences" what it means for a sequence to converge or diverge. We said that in order to determine whether a sequence $\{a_n\}$ converges or diverges, we need to examine its behaviour as n gets bigger and bigger. The way we do this is to calculate $\lim_{n\to\infty} a_n$. Before we look at an example we first state a theorem.

Theorem

Let L be a real number. Let f be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n\to\infty} a_n = L$

As we saw in the handout "Introduction to Sequences", a sequence can be thought of as a function of n. This means that we can use the methods we know for calculating the limit of a function of a real variable to find the limit of a sequence.

Example 1

In this example we want to determine if the sequence

$$\{a_n\} = \left\{\frac{n^2+2n+5}{2n^2+4n-2}\right\}$$

converges or diverges. Using the theorem above, if we let $f(x) = \frac{x^2+2x+5}{2x^2+4x-2}$ then $f(n) = \frac{n^2+2n+5}{2n^2+4n-2} = a_n$. In order to examine the sequence's behaviour as n gets bigger and bigger, we need to find $\lim_{n\to\infty} \frac{n^2+2n+5}{2n^2+4n-2}$.

We can now use standard methods (see handout"Limits to Infinity") to calculate the limit.

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We write

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2+2n+5}{2n^2+4n-2} &= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{5}{n^2}}{\frac{2n^2}{n^2}+\frac{4n}{n^2}-\frac{2}{n^2}} \\
&= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{5}{n^2}}{2+\frac{4}{n}-\frac{2}{n^2}} \\
&= \frac{1+0+0}{2+0-0} \\
&= \frac{1}{2}.
\end{aligned}$$

This tells us is that if we were to let n get bigger and bigger, $\{a_n\}$ would get closer and closer to $\frac{1}{2}$. In other words the sequence $\{a_n\}$ converges to $\frac{1}{2}$.

Example 2

Once again, we want to determine if the sequence

$$\{a_n\} = \left\{\frac{2n^3+5}{3n^2+1}\right\}$$

converges or diverges. We proceed as in the example above.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{2n^3+5}{3n^2+1} \\
&= \lim_{n\to\infty} \frac{\frac{2n^3}{n^3}+\frac{5}{n^3}}{\frac{3n^2}{n^3}+\frac{1}{n^3}} \\
&= \lim_{n\to\infty} \frac{2+\frac{5}{n^3}}{\frac{3}{n}+\frac{1}{n^3}}.
\end{aligned}$$

A little thought reveals that as n goes to infinity, the numerator goes to 2 and the denominator goes to 0. This means the $\{a_n\}$ will increase without bound. In other words $\lim_{n\to\infty} a_n = \infty$. We therefore conclude that the sequence $\{a_n\}$ diverges.

Try the following exercises for practice. Determine in each case if the sequence converges or diverges.

(a) $\{a_n\} = \left\{\frac{5n^3+n^2-6}{2n^3-4n^2-4}\right\}$ (b) $\{a_n\} = \left\{\frac{9n^2-n-3}{2n^3+4n^2+n-4}\right\}$ (c) $\{a_n\} = \left\{\frac{5n^3+6}{4n^2+n+11}\right\}$

Solutions

(a) Converges to $\frac{5}{2}$. (b) Converges to 0. (c) Diverges.