Eigenvalue equation for T:

$$Tf(r) = \int_0^r k(r,s)\, f(s)\, ds = \lambda f(r),$$

If $\lambda \neq 0$ then this is equivalent to

$$\frac{1}{\lambda} Tf = f \quad \text{fixed pt equation} \quad f = Qf$$

<u>Fixed pt thm</u> $Q : M \to M$ a contraction

(M metric space, metric d, M complete

$\exists \alpha < 1 \ \forall x, y \in M : d(Q(x), Q(y)) \leq \alpha\, d(x,y)$)

Then Q has a unique fixed pt ($x \in M$, $Qx = x$).

e.g. $\sqrt{2}$, $x^2 - 2 = 0$ $\qquad x = x - \frac{f(x)}{f'(x)} = x - \frac{x^2-2}{2x} = \frac{x}{2} + \frac{1}{x}$

$Q = \frac{1}{\lambda} T$ is a contraction if $\frac{\|T\|_{op}}{\lambda} < 1$ i.e.

for $\lambda > \|T\|_{op} = \sup_r \int_0^r |k(r,s)|\, ds < r \cdot \|k\|_\infty$

If $\lambda > \|k\|_\infty$ then λ is no eigenvalue.

$\underline{\text{ODE}}$ $\quad y' = \underset{C^1}{F}(y) \rightsquigarrow \quad y(t) = \int_0^t F(y(s))\, ds = Q_y(t)$

$$\left[Q(y) - Q(\tilde{y})\right](t) = \int_0^t F(y(s)) - F(\tilde{y}(s))\, ds$$

$$|\quad\downarrow\quad| \leq \int_0^t \|dF\|_\infty \cdot |y - \tilde{y}|(s)\, ds$$

$$\|Q(y) - Q(\tilde{y})\|_\infty \leq \underbrace{t \cdot \|dF\|_\infty}_{<1} \cdot \|y - \tilde{y}\|_\infty$$

Spectral Thm for cpct sa op:

$T \in \mathcal{L}(H, H)$, $\overline{T B_1^H(0)}$ cpct, $\underset{sa}{\langle Tx, y\rangle = \langle x, Ty\rangle}$

$\underline{\text{Then}}$ $\quad \operatorname{Spec} T$ consists of eigenvalues $\{\lambda_1, \lambda_2, \dots, \underset{\text{if } \dim H \text{ infinite}}{\underset{\uparrow}{0}}\}$

$$\|T\|_{op} = |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots \qquad \underset{\mathbb{R}}{\cap}$$

$$\lim_{i\to\infty} \lambda_i = 0, \quad \dim E(T, \lambda_i) < \infty$$

$$H = \bigoplus_{i=1}^{\infty\,\perp} E(T, \lambda_i) \overset{\perp}{\oplus} \underbrace{E(T, 0)}_{\ker T}$$

$\underline{\text{Thm}}$ $\quad (H = L^2 X) \qquad k \in L^2(X \times X)$

$$[Tf](x) = \int_X k(x,y)\, f(y)\, dy$$

T is cpct, $\|T\|_{op} \leq \|k\|_2$

T is sa if and only if $k(x,y) = \overline{k(y,x)}$ for almost all x,y

<u>proof</u>:

$$\langle f \mid Tg \rangle = \int_X f(x)\, \overline{Tg(x)}\, dx$$

$$= \int_X f(x) \overline{\int_X k(x,y)\, g(y)\, dy}\, dx$$

$$= \int_{X\times X} f(x)\, \overline{g(y)}\, \overline{k(x,y)}\, dy\, dx$$

$$= \int \left(\int k(y,x)\, f(x)\, dx \right) \overline{g(y)}\, dy$$

$$= \langle Tf, g \rangle$$

<u>For the converse</u>: Assume T selfadjoint, $\langle f, Tg \rangle = \langle Tf, g \rangle$

$$0 \overset{(*)}{=} \int_{X\times X} f(x)\, \overline{g(y)}\, \underbrace{\left(\overline{k(x,y)} - k(y,x)\right)}_{= 0 \text{ in } L^2} dx\, dy \quad \text{for all } f, g$$

X measure space i.e. a set X, a σ-algebra $\mathcal{A} \subset \mathcal{P}(X) = 2^X$,

measure $\mu : \mathcal{A} \longrightarrow \mathbb{R}_0^+$

σ-Algebra on $X \times X$ is the σ algebra generated by boxes i.e.

products $A_1 \times A_2$, $A_1, A_2 \in \mathcal{A}$

product measure: $\mu(A_1 \times A_2) = \mu(A_1) \cdot \mu(A_2)$

If $f = \chi_{A_1}$, $f(x) = \begin{cases} 0 & x \notin A_1 \\ 1 & x \in A_1 \end{cases}$, $g = \chi_{A_2}$

then $f \otimes g = \chi_{A_1 \times A_2}$ $\quad f \otimes g\,(x,y) = f(x) \cdot g(y)$

By $(*)$
$$\int_{A_1 \times A_2} \overline{k(x,y)} - k(y,x)\, dy\, dx = 0$$

$$\Rightarrow \overline{k(x,y)} - k(y,x) = 0 \text{ for almost all } x, y$$

Jordan Normal form: $A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ then there is a basis B for $\mathbb{C}^n$ s.th.

$$A_{BB} = \left(\begin{array}{cccc} \begin{matrix}\lambda_1 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_1\end{matrix} & & & 0 \\ & \boxed{\begin{matrix}\lambda_2 & 1 & 0\\ & \ddots & 1\\ 0 & & \lambda_2\end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix}\lambda_k & 1 & 0\\ & \ddots & 1\\ 0 & & \lambda_k\end{matrix}} \end{array}\right)$$

2 polynomials: $\text{charpol}_A(x) = \det(A - x\,\text{id})$

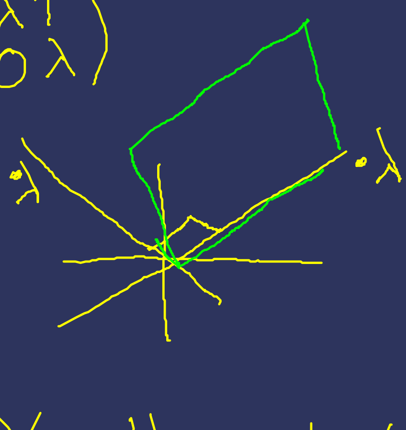
$\text{minpol}_A(x) = $ smallest monic polynomial $m \in \mathbb{C}[X]$ with $m(A) = 0$

If $f(A) = 0$ then $\text{minpol}_A \mid f$

Cayley Hamilton: $\text{charpol}_A(A) = 0$, hence

$\text{minpol}_A \mid \text{charpol}_A$

$\dim = 2$: $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$ $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$



$mp_{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} = (t-\lambda)$ $\quad$ $mp \overset{\text{if } \lambda \neq \mu}{=} chp = (\lambda - t)(\mu - t)$ $\quad$ $ch = (\lambda - t)^2 = mp$

$\dim = 3$:
$$\begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & \\ & \lambda & \\ & & \mu \end{pmatrix}, \quad \underbrace{\begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}}$$

$$A - \lambda = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$
$$(A-\lambda)^2 = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$$
$$(A-\lambda)^3 = 0$$

$1 = \lambda$

<u>Thm</u> X Banach, $k \in \mathcal{K}(X,X)$, $\boxed{T = 1 - k}$

$$N_m = \ker (1-k)^m$$
$$F_m = \operatorname{image} (1-k)^m$$

<u>Thm</u> $N_1 \subset N_2 \subset \dots N_m \subset N_{m+1} \subset \dots$

$$F_1 \supset F_2 \supset \dots F_m \supset F_{m+1}$$

$\dim N_m < \infty$, $\operatorname{codim} F_m < \infty$, N_m, F_m closed

$\exists\, m$, $N_m = N_{m+1} =: N$, $F_m = F_{m+1} =: F$, $\quad X = N \oplus F$

$$\circlearrowleft_{k} \quad \circlearrowleft_{k}$$

$$k = \left(\begin{array}{c|c} \begin{matrix} \ddots & 1 & m \\ & \ddots & 1 \\ 0 & & \ddots \end{matrix} & 0 \\ \hline 0 & k|_F \end{array}\right)$$

<u>Tychonov's Thm</u> The product of cpct spaces is cpct

<u>example</u> $I^I = \{ f : I \to I \} = \bigtimes_{j \in I} I_j$, product <u>top</u>

$$\begin{tikzcd} A\times B \arrow[r] \arrow[dr] & B \\ & A \end{tikzcd}$$

$$\begin{tikzcd} \bigtimes_j I_j \arrow[r, "pr_j"] \arrow[d, equal] & I_j \\ I^I \arrow[ur, "ev_j"] & \end{tikzcd} \qquad f \mapsto f(j)$$

product top is the smallest topology so that all projections are cts.

In the example I^I, evaluation are cts: For $U\subset I$ open,

$$\{f: I\to I \mid f(j)\subset U\} \text{ is open}$$
$$= ev_j^{-1}(U)$$

$f_n \xrightarrow[n\to\infty]{} f$ if $\forall j$ and $U\subset I$ open, $f(j)\in U$

we have $f_n(j)\in U$ for almost all n

pointwise convergence!

Tychonov: Every sequence in I^I has a ptwise cvgt subsequence.

Weak Topologies

maps induce topologies:

$$X \xrightarrow{f} (Y, \tau^Y)$$

$$\begin{array}{ccc} \text{gp, vsp, top} & & \\ M & \xrightarrow{f \text{ map}} & N \\ \text{set} & & \text{set} \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{K} \\ \text{top?} & \text{cts} & \end{array}$$

pull back top $f^* T^Y = \{ f^{-1}(V) \mid V \in T^Y \}$

initial top of f

$$= \bigcap_{T \text{ top on } X,\ f \text{ cts wrt } T} T$$

$(X, T^X) \xrightarrow{f} Y$ push forward topology, final top of f

$$f_* T^X = \{ V \subset Y \mid f^{-1} V \in T^X \}$$

Let X, Y be normed vector spaces

top X. $\quad n : X \longrightarrow \mathbb{R}^+_0$ cts but more:

$$B_b(0) - \overline{B_a(0)} = n^{-1}(a, b)$$

$$B_r(0) = n^{-1}[0, r)$$

$d : X \times X \to \mathbb{R}^+_0$, $(x, y) \mapsto n(x - y)$ cts

---

<u>Def</u> The weak topology on X is the initial top of all $f \in X'$ together, i.e. $\quad f : X \longrightarrow \mathbb{K}$

$X' = \mathcal{L}(X, \mathbb{K})$

the topology $\displaystyle\bigcap_{\substack{T \text{ top on } X \\ f \text{ cts wrt } T \text{ forall } f \in X'}} T$

$= \{ \text{unions of } f_1^{-1} U_1 \cap \dots \cap f_k^{-1} U_k \}$

$U_j \subset \mathbb{K}$ open

$f_j \in X'$

<u>Lemma</u> (X, wk) is Hausdorff

<u>proof</u> Let $a, b \in X$, $a \neq b$

By Hahn Banach, there is $f \in X'$, $f(a) \neq f(b)$

($f \in X'$, $f(a-b) \neq 0$ by extending

$\operatorname{span}\{a-b\} \longrightarrow \mathbb{K}$ to X

$\lambda(a-b) \longmapsto \lambda$)

Say $f(a-b) = 1$. Then

$$\underset{\ni a}{f^{-1}\left(B^{\mathbb{K}}_{\frac{1}{2}}(f(a))\right)} \cap \underset{\ni b}{f^{-1}\left(B^{\mathbb{K}}_{\frac{1}{2}}(f(b))\right)} = \emptyset$$

wk-open

<u>Lemma</u> $(x_n)_n \in X^{\mathbb{N}}$. Then $x_n \xrightarrow[n\to\infty]{} x$ wk

if and only if $\forall f \in X'$: $f(x_n) \xrightarrow[n\to\infty]{} f(x)$

<u>proof</u> If $x_n \overset{n\to\infty}{\underset{wk}{\longrightarrow}} x$ then $f(x_n) \xrightarrow[n\to\infty]{} f(x_n)$ by

continuity of f wrt wk.

Assume $\forall f \in X' : \underline{f(x_n) \xrightarrow[n\to\infty]{} f(x)}$.

Let $U \subset X$, U wk open, $x \in U$. Then there are $V_1 \dots V_k \subset \mathbb{K}$ open, $f_1, \dots, f_k \in X'$ such that

$$x \in \overbrace{f_1^{-1} V_1 \cap \dots \cap f_k^{-1} V_k}^{U_k} \subset U$$

Since $f_j x_n \xrightarrow[n\to\infty]{} f_j(x)$, almost all $f_j x_n \in f_j V_j$
(all but finitely many)

hence for almost all n

$$x_n \in U_k \subset U$$ $\boxtimes$

<u>Lemma</u> $x_n \xrightarrow[wk]{n\to\infty} x$ then $(x_n)_n$ is bdd.

<u>proof</u> $\forall f \in X' : f(x_n) \xrightarrow[n\to\infty]{} f(x)$, hence

$\forall f \in X' : \{ f(x_n) \mid n \in \mathbb{N}\}$ bdd

By UBP, $\{x_n \mid n \in \mathbb{N}\}$ bdd $\blacksquare$

wk convergence $\not\Rightarrow$ norm convergence

example: In $\ell^2 = \ell^2(\mathbb{N}) = \{a \in \mathbb{C}^{\mathbb{N}} \mid \sum |a_n|^2 < \infty\}$

let $e_n(k) = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases} = (0,0,\dots,0,\overset{n}{1},0,\dots)$

<u>Claim</u> $(e_n)_n$ does not contain a CS:

$$\|e_n - e_m\| = \begin{cases} 0 & n = m \\ \sqrt{2} & n \neq m \end{cases}$$

<u>Claim</u> $e_n \xrightarrow[wk]{n\to\infty} 0$: Let $\alpha \in (\ell^2)'$, By RRT

there is $a \in \ell^2$ st. $\alpha = \langle \cdot \,|\, a \rangle$

$$\alpha e_n = \langle e_n \,|\, a \rangle = e_n(1)\cdot\overline{a(1)} + e_n(2)\,\overline{a(2)} + \dots$$
$$= \overline{a(n)} \xrightarrow[n\to\infty]{} 0$$

$\boxtimes$

$$\mu(K) = 0$$
$$\|$$
$$(\gamma\delta_{x_0})(K) = \gamma \cdot [x_0 \in K]$$

<u>Lemma</u> $T \in \mathcal{L}(X,Y)$ [T linear, norm cts]

Then T wk-cts

$$? \longrightarrow f^{-1}(W) \xrightarrow{f^{-1}} W$$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[rr, bend right, "\text{cts}, \forall f"'] & Y \arrow[r, "f"] & \mathbb{K} \end{tikzcd}$$

<u>pf</u> $V \subset Y$ wk-open,

V = union of finite intersections of sets of the form

$f^{-1}(W)$, $W \subset \mathbb{K}$ open.

$$T^{-1}(f^{-1}W) = (f \circ T)^{-1}(W) \underset{\text{wk-open}}{\subset} X$$

because $f \circ T \in X'$ $\boxtimes$

---

weak-* topology

weakest | smallest top on X' making ev_x, $x \in X$ cts.

$$\begin{array}{ccc} C(X) & & C(X') \\ \cup & & \cup \\ X & X' & X'' = (X')' \\ & & \cup \\ & & \mathrm{ev} \end{array}$$

$X \xrightarrow{\mathrm{ev}} X''$

norm $\supset$ wk (on X)

norm $\supset$ wk $\supset$ wk* (on X')

$f(x) = \mathrm{ev}_x(f)$

$\mathrm{ev}_X = \{\mathrm{ev}_x \mid x \in X\}$

$\mathrm{ev}_X \cong X$

$$\text{weak * top on } X' = \bigcap_{\substack{T \text{ top on } X' \\ \forall x \in X,\ \mathrm{ev}_x : X' \to \mathbb{K} \text{ is cts} \\ f \mapsto f(x)}} T$$

$f \in X'$

$$\|f\|_{op} = \sup\left\{\frac{|fx|}{\|x\|} \;\middle|\; x \in X,\ x \neq 0\right\}$$

$= \{$ unions of finite intersections of sets of the form

$\mathrm{ev}_x^{-1} V$, $x \in X$, $V \subset \mathbb{K}$ open $\}$

$$\mathrm{ev}_x^{-1} V = \{f \in X' \mid \mathrm{ev}_x f = f(x) \in V\}$$

wk-* $\subset$ wk $\subset$ norm top on X'

Lemma wk* is Hausdorff

proof If $f_1, f_2 \in X'$, $f_1 \neq f_2$ then

there is $x \in X$, $f_1(x) \neq f_2(x)$. wlog,

$(f_1 - f_2)x = 1$ $\quad$ $ev_x f_1 - ev_x f_2$

$$\underbrace{ev_x^{-1}(B^{\mathbb{K}}_{1/2} f_1(x))}_{\ni f_1} \cap \underbrace{ev_x^{-1}(B^{\mathbb{K}}_{1/2} f_2(x))}_{\ni f_2} = \emptyset$$

wk* open

$\boxtimes$

Lemma wk* convergence is pointwise convergence

$(f_n)_n \in (X')^{\mathbb{N}}$,

$$f_n \xrightarrow[n\to\infty]{} f \text{ wk*} \iff \forall x \in X : f_n(x) \xrightarrow[n\to\infty]{} f(x)$$

proof "$\Rightarrow$" $x \in X$, $ev_x : X' \to \mathbb{K}$ is cts wrt wk-* top on X'

(by def of wk-*)

$$f_n \xrightarrow[n\to\infty]{wk*} f \Rightarrow \underset{\substack{\| \\ f_n(x)}}{ev_x f_n} \xrightarrow[n\to\infty]{} \underset{\substack{\| \\ f(x)}}{ev_x f}$$

„$\Leftarrow$" assume $f_n \xrightarrow[n\to\infty]{} f$ ptwise. To prove wk-* convergence,

let $U \subset X'$, $f \in U$, U wk-* open. Then there are

$x_1 \dots x_k \in X$, $V_1 \dots V_k \subset \mathbb{K}$ open, so that

$$f \in ev_{x_1}^{-1} V_1 \cap ev_{x_2}^{-1} V_2 \cap \dots \cap ev_{x_k}^{-1} V_k \subset U$$

$$f_n(x_j) \xrightarrow[n\to\infty]{} f(x_j) \in V_j \quad \text{for all } j = 1 \dots k$$

hence $f_n(x_j) \in V_j$ for almost all $n \in \mathbb{N}$

$$N_j = \{ n \mid f_n(x_j) \notin V_j \} \text{ is finite for all } j$$

$$\{ n \mid \underbrace{f_n(x_j)}_{ev_{x_j}(f_n)} \notin V_j \text{ for some } j = 1 \dots k \} = \bigcup_{j=1}^{k} N_j \quad \text{finite}$$

$$\{ n \mid f_n \notin \underbrace{ev_{x_1}^{-1} V_1 \cap ev_{x_2}^{-1} V_2 \cap \dots \cap ev_{x_k}^{-1} V_k}_{\subset U} \} \quad \text{finite}$$

$\boxtimes$

$$y_n \to y \iff \forall \underset{\substack{\text{open}\\ \text{nbhd}}}{U} \ni y : \{ n \mid y_n \notin U \} \text{ finite}$$

X top space. X is separable if

$$\exists M \subset X \text{ dense}, \ M \text{ countable}$$

example: $\mathbb{R} \supset \mathbb{Q}$. $\ell^2(\mathbb{N}) \supset \ell^2(\mathbb{N}) \cap \mathbb{Q}^{\mathbb{N}}$

$$\supset \bigcup_r \underbrace{T_r(\ell^2\mathbb{N} \cap \mathbb{Q}^{\mathbb{N}})}_{=\mathbb{Q}^r,\ \text{countable}}$$ is a countable union of ctble sets.

dense

$$T_r(a_1, a_2, \dots a_r\ a_{r+1}, \dots) = (a_1 \dots a_r\ 0000\dots)$$

$$T_r a \xrightarrow[r\to\infty]{\|\ \|_2} a \ : \|a - T_r a\|_2^2 = |a_{r+1}|^2 + |a_{r+2}|^2 + \dots \xrightarrow[r\to\infty]{} 0$$

$\Rightarrow \ell^2\mathbb{N}$ is separable

$L^2([0,1])$ has a countable ONB, Fouriertrafo
$\cong$ $(t \mapsto e^{2\pi i k t}, k \in \mathbb{Z})$
$\ell^2\mathbb{N}$

$$L^2(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} \mid \int_{\mathbb{R}^n} |f|^2 < \infty \} \Big/ \{ f \mid \int |f|^2 = 0 \}$$

$\cong$

$\ell^2\mathbb{N}$ hint: $\mathbb{R}^n = \bigcup_{\nu=1}^{\infty} [-\nu,\nu]^n$, use

truncation: $L^2\mathbb{R}^n \longrightarrow L^2([-\nu,\nu]^n)$

$$f \longmapsto f|_{[-\nu,\nu]^n}$$

(X,T) is 1st countable if $\forall x \in X \ \exists \, \mathcal{B}_x \subset T$:

$$\forall U \in \mathcal{B}_x : x \in U$$

$\mathcal{B}_x$ countable

$$\forall V \in T, x \in V \ \exists \, U \in \mathcal{B}_x : U \subset V$$

"countable neighbourhood basis"

(X,T) is 2nd countable if T has a countable basis. ($\mathcal{B} \subset T$ countable,

$$\forall V \in T : V = \bigcup_{U \in \mathcal{B}, U \subset V} U \quad)$$

<u>examples</u> metric spaces are 1st countable

$$\mathcal{B}_x = \{ B_{1/n}(x) \mid n \in \mathbb{N} \}$$

n so that $\frac{1}{n} < r$

$B_{1/n}(x)$

$\mathbb{R}^n$ is 2nd countable:

$$\mathcal{B} = \{ B_{1/n}(q) \mid n \in \mathbb{N}, q \in \mathbb{Q}^n \}$$

<u>Thm</u> If Z is a 1st countable compact space, then

Z is sequentially cpct.

$\overline{B_1^{X'}(0)}$ wk-$*$ cpct (Alaoglou)

<u>Thm</u> $B_1^{X'}(0)$ wk-$*$ is 1st countable if X is sep

$$B_1^{X'}(0) = \{f \in X' \mid \|f\|_{op} < 1\}$$

<u>proof</u> Let $M \subset X$ be countable, dense.

$$U(f, m_1 \ldots m_k, \tfrac{1}{n}) = ev_{m_1}^{-1}\left(B_{1/n}(f(m_1))\right) \cap \ldots \cap ev_{m_k}^{-1}\left(B_{1/n}(f(m_k))\right)$$

$$= \{g \mid |g(m_1) - f(m_1)| < \tfrac{1}{n}, \ldots \underline{|g(m_k) - f(m_k)| < \tfrac{1}{n}}\}$$

$$\subset U(f, x_1 \ldots x_k, \varepsilon) = \{g \mid |g(x_j) - f(x_j)| < \varepsilon\}$$

if $\|m_i - x_i\| < \frac{1}{3}\varepsilon \qquad \frac{1}{n} < \frac{1}{3}\varepsilon$

$$|g(x_j) - f(x_j)| = |g(x_j) - g(m_j) + f(m_j) - f(x_j) + g(m_j) - f(m_j)|$$

$$\leq |g(x_j) - g(m_j)| + |f(m_j) - f(x_j)| + |g(m_j) - f(m_j)|$$

$$\leq 2\|x_j - m_j\| + \|g(m_j) - f(m_j)\| \overset{!}{<} \varepsilon$$

$\quad < 2\delta \qquad\qquad \leq \frac{1}{n}$

$\boxtimes$

Riesz Representation Thm

X locally cpct top space, $C(X) = \{\text{cts } f: X \to \mathbb{C}\}$

$C_b(X) = \{f \in C(X) \text{ bdd}\}$ $\quad \|f\|_\infty = \sup |f| \in [0,\infty]$

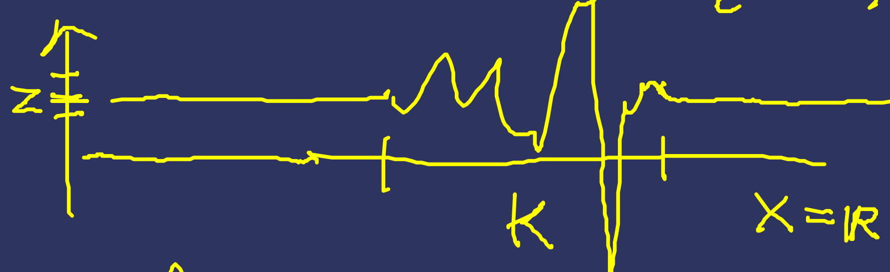
$\cup$

$C_\infty(X) = \{f \in C(X) \text{ (bdd)}$ $\quad$ complete

$\forall \varepsilon\ \exists\ K \subset X$ cpct, $z \in \mathbb{C}: f(X \setminus K) \in B_\varepsilon(z)\}$

$\lim_{x \to \infty} f$ is def'd for $f \in C_\infty$

have "value at ∞"



$$\begin{tikzcd} f: X \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{C} \\ X \cup \infty = X^+ \arrow[ur] & \end{tikzcd}$$

1-pt-compactification

$\to C_0(X) = \{f \in C(X) \text{ (bdd)}, \forall \varepsilon\ \exists K \subset X_{\text{cpct}}: f(X \setminus K) \subset B_\varepsilon(0)\}$

$= \text{Ban lim}$

$$c_0 \subset c \subset b$$

$$\downarrow \lim \qquad \cancel{\downarrow}\ \widehat{\lim} \notin \ell^1$$

$$\ell^1 \longrightarrow b'$$

$$a \longmapsto \underbrace{\left(x \mapsto \sum_{n=1}^{\infty} a_n x_n\right)}_{\phi(a)}$$

$$\widehat{\lim} \notin \phi(\ell)$$

$$C_0(X)' \cong M(X)$$

regular Borel measures

$$X \xrightarrow{ev} C_0(X)'$$

$$x \longmapsto \underbrace{(f \longmapsto f(x))}_{ev_x}$$

corresponds to pt measure

$$ev_x \longleftrightarrow \delta_x$$

$$\delta_x(A) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

On $X = \mathbb{R}$ $\quad \lambda([a\,b]) = b - a$ Lebesgue measure

$\qquad\qquad\qquad\quad (\ \ \ )$

<u>Def</u> X a set, $\Omega \subset \mathcal{P}X$ a σ-algebra ($X, \phi \in \Omega$, complements, ctble $\cup, \cap$ stay in Ω)

A (signed, $\mathbb{C}$-valued) measure is a countably additive function $\mu : \Omega \longrightarrow \mathbb{R}_0^+ \ (\cup\{\infty\}) \mid \mathbb{R} \mid \mathbb{C}$

$$A_i \in \Omega,\ A_i \cap A_j = \phi \quad \mu\left(\bigcup A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$
$i \in \mathbb{N}$

<u>Hahn Jordan decomposition Thm</u>

(X, Ω, μ) measure space, $\mu : \Omega \to \mathbb{R}$

Then $\exists \; X^+, X^- \in \Omega$, $X^+ \dot\cup X^- = X$

$$\mu(A) = \underbrace{\mu(A \cap X^+)}_{\geq 0} - \big(\underbrace{-\mu(A \cap X^{-1})}_{\leq 0}\big)$$

$\mu = \mu_+ - \mu_-$, μ_+, μ_- positive measures.

proof if $V \subset X^+$, then $\mu(V) \geq 0$

X^+ is maximal with this property

to prove existence of X^+, apply Zorn's Lemma to

$\{ Q \in \Omega \mid \forall V \subset Q, \mu(V) \geq 0 \}$,

PO: $Q_1 < Q_2$ if $Q_1 \subset Q_2$

If $\{Q_i\}_i$ is a chain, then $\bigcup_i Q_i$ is upper bound

ZL $\rightsquigarrow X_+$

For X_- use

$\{ P \in \Omega \mid \forall V \subset P, \mu(V) \leq 0, P \subset \overbrace{X \setminus X_+}^{X_-} \}$

$\boxtimes$

μ signed $\quad \mu = \mu_+ - \mu_-$

μ complex valued: $\quad \mathrm{Re}\,\mu \;,\; (\mathrm{Re}\,\mu)(A) = \mathrm{Re}(\mu(A))$

$\mathrm{Im}\,\mu$

are signed measures.

$$\mu = \mathrm{Re}\,\mu^+ - \mathrm{Re}\,\mu^- + i\left(\mathrm{Im}\,\mu^+ - \mathrm{Im}\,\mu^-\right)$$

$$X = X_r^+ \,\dot\cup\, X_r^- = X_i^+ \,\dot\cup\, X_i^-$$

<u>Lebesgue Integral</u> $\quad f : X \to \mathbb{R}$ measurable

$f = f^+ - f^-$ $\qquad (f^{-1}(\text{Interval}) \in \Omega)$

$$\int_X f \, d\mu := \int_X f^+ d\mu - \int_X f^- d\mu$$

positive

Leb. integrable $\Rightarrow$ || $\quad \sup\left\{ \int_X s \, d\mu \;\middle|\; s \leq f^+ \;\; s \text{ stepfunction} \right\}$

$\inf\left\{ \int_X s \, d\mu \;\middle|\; s \geq f^+ \right\}$

$\forall A \in \Omega,$

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}, \quad \int_X 1_A \, d\mu = \mu(A)$$

stepfunction = linear comb of 1_A's $A \in \Omega$

Def. X top space, locally cpct

Borel σ-algebra is the σ-alg generated by the top.

(cts open, closed sets, ctble unions, $\cap$ of them, e.g.)

$$M(X) = \{\text{regular Borel measures on } X\}$$

μ regular if a) K cpct $\Rightarrow \mu(K) < \infty$

b) $A \subset \Omega \Rightarrow$

$$\mu(A) = \sup\{\mu(K) \mid K \subset A\}$$

$$= \inf\{\mu(U) \mid \underset{\text{open}}{U} \supset A\}$$

surjectivity?

RRT $\quad C_0(X)' \cong M(X) \leftarrow$ norm?

$$\left(\int_X f\,d\mu \longleftarrow f\right) \longleftarrow \mu$$

is an isometry:

Norm on $\mathcal{M}(X) \ni \mu$ $\qquad (a_n) \longrightarrow \sum |a_n|$

If $\mu = \mu_+ - \mu_-$, real valued, signed

$$\|\mu\| = \mu_+(X) + \mu_-(X)$$

If μ is complex valued :

$|\mu|(A)$ = variation of $\mu_{\text{on } A}$ $= \sup \left\{ \sum |\mu(E_i)| \;\middle|\; A = \dot{\bigcup_i} E_i \right\}$

$$\|\mu\| = |\mu|(X)$$