For those who have a lot of outstanding homework, please hand up problems 10, 12, 13, 15, 27 with priority. Review problems 26 and 28.

The current homework is on the last page.

1. Simplify the following
   \[ \{ r \in \mathbb{R} \mid \exists z \in \mathbb{Z} : zr \in \mathbb{Z} \} \]
   \[ \text{Solution: } \mathbb{R} \]
   Note that \( \{ r \in \mathbb{R} \mid \exists z \in \mathbb{Z} \setminus \{0\} : zr \in \mathbb{Z} \} = \mathbb{Q} \)
   \[ \{ x \in \mathbb{R} \mid \forall y \in \mathbb{R} \exists z \in \mathbb{R} : x > z + y \} \]
   \[ \text{Solution: } \mathbb{R} \]
   \[ \bigcap_{n \in \mathbb{N}} \{ x^2 \mid 0 \leq nx \leq \sup \{ u \mid u \in \mathbb{R} \& u^2 < n \} \} \]
   \[ \text{Solution: } = \{ 0 \} \]
   \[ \text{Hint: glossary:} \]
   \[ \exists \text{ there is} \]
   \[ \forall \text{ for all} \]
   \[ : \text{ such that} \]
   \[ \& \text{ and} \]
   \[ \mathbb{N} = \{1, 2, 3, 4, \ldots \} \text{ set of natural numbers excluding 0} \]
   \[ \mathbb{R} \text{ set of real numbers} \]
   \[ \mathbb{Z} \text{ set of integers} \]

2. Recall that a topology on a set \( X \) is a subset \( T \subset \mathcal{P}(X) \) of the power set of \( X \) such that \( T \)
   \[ (a) \theta, X \in T , \]
   \[ (b) S \subset T \implies \bigcup_{U \in S} U \in T \text{ and} \]
   \[ (c) S \subset T \text{ finite } \implies \bigcap_{U \in S} U \in T . \]

For each of the following subsets of \( \mathcal{P}(\mathbb{R}) \) say whether it is a topology on \( \mathbb{R} \). If yes, say what it means for a sequence to converge with respect to this topology. If no show by a counterexample that at least one of the above properties \( (2a)-(2c) \) is violated.

(a) \( T_a = \{ [-a,a] \mid a \in \mathbb{R} \} \cup \{ \emptyset, X \} \)
   \[ \text{Solution: violates } \]
   \[ \forall n \in \mathbb{N} : [-1 + 1/n, 1 - 1/n] \in T_a \text{ but } \bigcup_{n \in \mathbb{N}} [-1 + 1/n, 1 - 1/n] = (-1, 1) \notin T_a \]

(b) \( T_b = \{ (a,a) \mid a \in \mathbb{R} \} \cup \{ \emptyset, X \} \)
   \[ \text{Solution: This is a topology. A sequence } (x_n)_{n \in \mathbb{N}} \text{ converges to } a \in \mathbb{R} \text{ with respect to this topology if} \]
   \[ \limsup_{n \to \infty} |x_n| = |a| \]
   \[ (c) T_c = \{ [0,n] \mid n \in \mathbb{N}_0 \} \cup \{ \emptyset, X \} \]
   \[ \text{Solution: violates } \]
   \[ \forall n \in \mathbb{N} : [0,n] \in T_c \text{ but } \bigcup_{n \in \mathbb{N}} [0,n] = [0, \infty) = \mathbb{R}_0^+ \notin T_c \]

(d) \( T_d = \{ [-n,n] \mid n \in \mathbb{N}_0 \} \cup \{ \emptyset, X \} \)
   \[ \text{Solution: This is a topology. If } k \in \mathbb{N}_0 \text{ and } k + 1 \geq |a| > k \text{ then a sequence } (x_n)_{n \in \mathbb{N}} \text{ converges to } a \text{ if } |x_n| \leq k + 1 \text{ eventually, i.e. if there is } n_0 \text{ such that } |x_n| \leq k + 1 \text{ for all } n > n_0. \]
3. List all topologies on the set \( X = \{0,1\} \).

**Solution:**

(a) trivial topology: \( \{\}, \{0,1\} \),

(b) discrete topology: \( \mathcal{P}(\{0,1\}) = \{\}, \{0\}, \{1\}, \{0,1\} \),

(c) \( \{\}, \{0\}, \{0,1\} \),

(d) \( \{\}, \{1\}, \{0,1\} \)

4. Let \( X \) be a set. Recall that a **basis for a topology** on \( X \) is a subset \( B \subseteq \mathcal{P}(X) \) such that

\[
\bigcup_{B \in B} B = X \quad \text{and} \quad \forall B_1, B_2 \in B, x \in B_1 \cap B_2 \exists B(x, B_1, B_2) \in B : x \in B(x, B_1, B_2) \subset B_1 \cap B_2
\]

and that the **topology generated by** \( B \) is

\[
T(B) = \left\{ \bigcup_{B \in Q} B \mid Q \subseteq B \right\}.
\]

For a finite set \( A \) we denote by \( \# A \in \mathbb{N}_0 \) the number of elements of \( A \).

(a) Let \( B \) be finite and a basis for a topology on a set \( X \). Find an upper bound \( b(\#B) \) on \( \#T(B) \) in terms of \( \#B \).

More precisely, you have to find a function \( b : \mathbb{N}_0 \to \mathbb{N}_0 \) such that for all sets \( X \) and all finite subsets \( B \subseteq \mathcal{P}(X) \) which are a basis for a topology on \( X \), we have \( \#T(B) \leq b(\#B) \).

**Solution:** By definition, any element of \( T(B) \) is of the form \( \bigcup_{B \in Q} B \). Thus we have a surjective map

\[
\mathcal{P}(B) \to T(B),
Q \mapsto \bigcup_{B \in Q} B
\]

and in particular, \( \#T(B) \leq \#\mathcal{P}(B) = 2^\#B \). Thus the function

\[
b : \mathbb{N}_0 \to \mathbb{N}_0 \quad \text{with} \quad b(n) = 2^n
\]

does the job.

(b) Prove that your bound is sharp, i.e. for any \( n \in \mathbb{N}_0 \) find a set \( X \) and a basis \( B \) for a topology on \( X \) such that \( \#B = n \) and \( \#T(B) = b(n) \).

**Solution:** For \( n \in \mathbb{N}_0 \) look at the discrete topology \( T = \mathcal{P}(X_n) \) on the set \( X_n = \{ k \in \mathbb{N} \mid k \leq n \} = \{1,2,3,\ldots,n\} \).

A basis for this topology is

\[
B_n = \{ \{k\} \mid k \in \mathbb{N}, k \leq n \} = \{\{1\},\{2\},\{3\},\ldots,\{n\}\}.
\]

Clearly, \( \#B_n = \#X_n = n \) and \( \#T(B) = \#\mathcal{P}(X_n) = 2^n \).

5. Let \( f : X \to Y \) be a map, \( X,Y \) sets. Prove or disprove by a counterexample:

(a) If \( T \) is a topology on \( Y \) then

\[
f^*(T) := \{ f^{-1}(V) \mid V \in T \}
\]

is a topology on \( X \).

**Solution:** This is true. Proof: Clearly, \( \emptyset = f^{-1}(\emptyset) \in f^*(T) \) and \( X = f^{-1}(Y) \in f^*(T) \). If \( U \subset f^*(T) \), then there is \( V \subset T \) such that

\[
U = \left\{ f^{-1}(V) \mid V \in V \right\}, \quad \text{hence}
\]

\[
\bigcup_{U \in U} U = \bigcup_{V \in V} f^{-1}(V) = f^{-1} \left( \bigcup_{V \in V} (V) \right) \in f^*(T).
\]

Finally, if \( U \subset f^*(T) \) is finite, then there is \( V \subset T \) finite such that

\[
U = \left\{ f^{-1}(V) \mid V \in V \right\}, \quad \text{hence}
\]

\[
\bigcap_{U \in U} U = \bigcap_{V \in V} f^{-1}(V) = f^{-1} \left( \bigcap_{V \in V} (V) \right) \in f^*(T).
\]
(b) If \( T \) is a topology on \( X \) and if \( f \) is surjective then
\[
f_*(T) := \{ f(V) \mid V \in T \}
\]
is a topology on \( Y \).

**Solution:** This only violates the intersection property, since generally
\[
f(U_1 \cap U_2) \subset f(U_1) \cap f(U_2)
\]
but equality needs not hold. A counterexample here is given by
\[
X = \{1, 2, 3, 4\}, \quad T = 0, X, \{1, 2\}, \{3, 4\}, \quad Y = \{1, b, 4\}
\]
and
\[
f: X \to Y \quad \text{is such that} \quad f(1) = 1, f(4) = 4, f(2) = f(3) = b.
\]

Then
\[
f_*(T) = \{ f(V) \mid V \in T \} = \{ \{\}, \{1, b, 4\}, \{1, b\}, \{b, 4\}\}
\]
which is not a topology since it does not contain \( \{1, b\} \cap \{b, 4\} = \{b\}\).

**Hint:** \( f(\emptyset) = \emptyset = f^{-1}(\emptyset) \). "\( f \) is surjective" means that \( f(X) = Y \).

**Please hand up 3-5 Monday, 22/02/2010 in class**

6. Recall that a topological space \((X, T)\) is connected if
\[
U, V \in T \setminus \{\emptyset\}, \quad U \cup V = X \implies U \cap V \neq \emptyset.
\]

If \((X, T)\) and \((Y, S)\) are topological spaces then a map \( f: X \to Y \) is continuous (with respect to the topologies \( T \) and \( S \)) if \( f^{-1}S \subset T \) (see problem 5a). A continuous map \( f: X \to Y \) is a homeomorphism if \( f \) is bijective and if the inverse \( f^{-1}: Y \to X \) is also continuous.

Let \( X, Y \) be topological spaces, \( X \) connected, and \( f: X \to Y \) be a surjective continuous map. Prove that \( Y \) is connected.

**Solution:** Assume \( U, V \subset Y \) are open nonempty and such that \( U \cup V = Y \). Since \( f \) is surjective the sets \( f^{-1}(U), f^{-1}(V) \) are also nonempty and since \( f \) is continuous these sets are open subsets of \( X \). We also have
\[
f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X.
\]

Since \( X \) is connected we infer that \( f^{-1}(U) \cap f^{-1}(V) \neq \emptyset \), say \( x \in f^{-1}(U) \cap f^{-1}(V) \). But then we have \( f(x) \in U \) and \( f(x) \in V \), hence \( f(x) \in U \cap V \).

7. Recall that a metric on a set \( X \) is a function \( d: X \times X \to \mathbb{R}_+^+ \) such that
\[
\forall x, y, z \in X: d(x, y) = d(y, x), d(x, z) \leq d(x, y) + d(y, z), d(x, y) = 0 \iff x = y.
\]

If \( d \) is a metric on a set \( X \) then the topology \( T(d) \) induced by \( d \) is the topology generated by the set of balls \( \text{wrt} \ d \), i.e.
\[
T(d) = T(\{ B_r(x) \mid r \in \mathbb{R}_+, x \in X \})
\]
(see :ref:`metric topologies` for the notation on the right hand side here). Which of the following topologies \( T_i \) on \( X = \mathbb{R} \) is induced from a metric on \( \mathbb{R} \)?

(a) \( T_a = T(\{(z, b) \mid z, b \in \mathbb{Z}\}) \)

**Solution:** This is not metric: Let \( x \in (0, 1] \) Then if \( x \in U \subset T_a \) we must have \((0, 1] \subset U\). Assume \( d \) were a metric with \( T_a = T(d) \). Let \( p \in (0, 1], p \neq x, \) and \( r = d(p, x)/2 \). Then \( B_r(x) \in T_a = T(d) \), hence \( p \in (0, 1] \subset B_r(x) \). But since \( d(p, x) = 2r > r \) we have \( p \notin B_r(x) \), hence \( p \notin (0, 1], \) a contradiction.

(b) \( T_b = \mathcal{P}(\mathbb{R}) \)

**Solution:** This is a metric topology. Define the distance function \( d: X \times X \to \mathbb{R}_+^+ \) by \( d(x, y) = 1 \) if \( x \neq y \) and 0 if \( x = y \). For \( x \in X \) we have \( B_r(x) = \{ x \} \) if \( 0 < r \leq 1 \) and \( B_r(x) = X \) if \( 1 < r \). Thus a basis for the metric topology is
\[
B = \{ B_r(x) \mid r \in \mathbb{R}_+, x \in X \} = \{ X \} \cup \{ \{ x \} \mid x \in X \}.
\]

But this basis generates the discrete topology
\[
T(B) = \left\{ \bigcup_{U \in Q} U \mid Q \subset B \right\} = \mathcal{P}(X).
\]

(c) \( T_c = \{ (a, b) \mid a, b \in \mathbb{R} \cup \{ \pm \infty \}, -\infty \leq a < 0 < b \leq +\infty \} \cup \{\emptyset\} \)

**Solution:** This is not metric by a reason similar to that in :ref:`metric topologies`. If \( \emptyset \neq U \subset T_c \) then \( 0 \in U \). Assuming that \( d \) is a metric with \( T(d) = T_c \) let \( r = d(0, 1)/2 \). Then \( 0 \notin U \) if \( B_r(1) \in T(d) = T_c \), a contradiction.

**Hint:** Either give a metric \( d \) so that \( T_i = T(d) \) or prove that \( T_i \) does not have a property which a metric topology necessarily has.
8. If \((X,T)\) is a topological space and \(A \subset X\) then the subspace topology on \(A\) is
\[
T|_A := \{ A \cap U | U \in T \}
\]
Prove that there is no homeomorphism \(f: [0,1) \to (0,1)\) (where both intervals carry the subspace topology from \(\mathbb{R}\)).

**Solution:** Assume \(f: [0,1) \to (0,1)\) is a homeomorphism. Then so is the restriction
\[
f|_{(0,1)}: (0,1) \to (0,1) \setminus \{f(0)\}
\]
But \((0,1)\) is connected and \((0,1) \setminus \{p\}\) is not for any \(p \in (0,1)\). Hence \((0,1) \setminus \{f(0)\}\) cannot be the image of \((0,1)\) under a continuous map by \(\square\)

**please hand up 6-8 Monday, 1/3/2010 in class**

9. Let \((X,T)\) be a topological space.

Assume \(f\) is continuous wrt \(T\), \(U\) is open (wrt \(T\)), and \(V\) is somehow (how?) related to \(B^X\), or, in the spirit of problem \(\square\) if and only if for all \(x \in X\), \(f\) is continuous at \(x\).

10. Find (formulas for) homeomorphisms:

\[
\mathbb{R}^n \xrightarrow{\approx} D^n \xrightarrow{\approx} S^n \setminus \{(0,\ldots,0,-1)\}
\]

where \(\mathbb{R}^n\) carries the standard topology and

\[
D^n = \left\{ x \in \mathbb{R}^n \big| \|x\|^2 = \sum_{i=1}^{n} x_i^2 < 1 \right\}
\]

and

\[
S^n = \left\{ x \in \mathbb{R}^{n+1} \big| \|x\|^2 = \sum_{i=1}^{n+1} x_i^2 = 1 \right\}
\]

are endowed with the subset topologies.

11. Let \((X,T^X)\) and \((Y,T^Y)\) be topological spaces and assume that \(B^X, B^Y\) are basis for the respective topologies, i.e.

\[
T^X = T(B^X) \quad \text{and} \quad T^Y = T(B^Y)
\]

Say what it means for a map \(f: X \to Y\) to be continuous wrt \(T^X, T^Y\), without reference to the topologies, only referring to the bases \(B^X, B^Y\).

**Hint:** Try something like

“\(f\) is continuous wrt \(T(B^X), T(B^Y)\), if for any \(V \in B^Y\) we have that \(f^{-1}(V)\) is somehow (how?) related to \(B^X\),

or, in the spirit of problem \(\square\)

“\(f\) is continuous wrt \(T(B^X), T(B^Y)\), if for any \(x \in X\) and \(B \in B^Y\), \(f(x) \in B\) ...”.

12. Say which of the following is true. If no, provide a counterexample.

(a) If \((X,T^X)\) and \((Y,T^Y)\) are topological spaces, then \(T^X \times T^Y\) is a topology on \(X \times Y\).

(b) If \(X\) is a set and \(A\) is a set of topologies on \(X\), then \(\bigcup_{T \in A} T\) is a topology on \(X\).

(c) If \(X\) is a set and \(A\) is a set of topologies on \(X\), then \(\bigcap_{T \in A} T\) is a topology on \(X\).

(d) A subset of a topological space is either open or closed.

(e) The intersection of any family of closed sets of a topological space is closed.

(f) Let \(T, T'\) be topologies on a set \(X\) such that \(T \subseteq T'\).

i. If \((X,T)\) is connected then \((X,T')\) is connected.

ii. If \((X,T')\) is connected then \((X,T)\) is connected.

**Hint:** Recall that if \((X,T)\) is a topological space, then a set \(U \subset X\) is called open (wrt \(T\)) if \(U \in T\), and a set \(A \subset X\) is called closed (wrt \(T\)) if \((X \setminus A) \in T\). These notions only make sense if some underlying topology is assumed.

If you think the statement is true, you need not provide a proof!

**please hand up 9-12 Monday, 8/3/2010 in class**

13. A topological space \((X,T)\) is Hausdorff if

\[
\forall x, y \in X, x \neq y \exists U, V \in T : x \in U, y \in V, U \cap V = \emptyset.
\]

Prove that a metric space \((X,d)\) is Hausdorff (i.e. that \((X,T(d))\) is Hausdorff).

14. Prove that a topological space \(X\) is Hausdorff if

\[
\Delta X = \{(x,x) | x \in X\} \subset X \times X
\]

is a closed subset in \(X \times X\) (wrt the product topology).
17. Consider the topology $T$.

16. Find a subset of $X$, $T$.

15. A topological space $X$ is path connected if for any two points $x, y \in X$ there is a path (i.e., a continuous map from an interval) $c: [0, 1] \to X$ with $c(0) = x, c(1) = y$. Prove that a path connected topological space is connected (see problem 5).

Hint: You might want to use the Intermediate Value Theorem, in the form stating that the interval $[0, 1]$ is connected.

16. Find a subset of $\mathbb{R}^2$ which is connected but not path connected.

17. Consider the topology $T = T(B)$ on $\mathbb{R} \cup \{\infty\}$ generated by

$$B = \{ (a, b) | a, b, \in \mathbb{R} \} \cup \{ \{\infty\} \cup \mathbb{R} \setminus [a, b] | a, b, \in \mathbb{R} \}.$$ 

Prove that $(\mathbb{R} \cup \{\infty\}, T)$ is homeomorphic with $S^1 = \{(x, y) | x, y, \in \mathbb{R}, x^2 + y^2 = 1\}$.

please hand up 13-17 Monday, 22/3/2010 in class

18. Let $(X, T)$ be a Hausdorff space and $\infty \not\in X$. The one point compactification of $(X, T)$ is the topological space

$$(X^*, T^*) = (X \cup \{\infty\}, T \cup \{\infty\} \cup X \setminus K | K \subset X \text{ compact})$$

Prove that the one point compactification of a Hausdorff space is compact.

Solution: Let $U \subset T^*$ be an open covering of $X$. Since $U$ covers $X^* \supset \infty$, there must be $V \in U, \infty \in V$, hence $V \not\in T$ and therefore a $K \subset X$ compact with $V = \{\infty\} \cup X \setminus K$. Thus

$$X^* = V \cup K = \bigcup_{U \in U} U.$$ 

In particular

$$K \subset \bigcup_{U \in U} U \setminus \{\infty\}$$

and therefore

$$\{U \setminus \{\infty\} | U \in U\}$$

is an open covering of $K$. By compactness, there is a finite subset $F \subset U$ such that

$$K \subset \bigcup_{U \in F} U \setminus \{\infty\}$$

hence

$$X^* = V \cup K = V \cup \bigcup_{U \in F} U \setminus \{\infty\} = V \cup \bigcup_{U \in F} U$$

which shows that $\{V\} \cup F \subset U$ is a finite subcovering of $U$ for $X^*$.

19. A continuous map $f: X \to Y$ is an embedding if $f: X \to \text{im } f$ is a homeomorphism (where im $f$ carries the subspace topology from $Y$). Recall that $f: X \to Y$ is a homeomorphism if $f$ has a continuous inverse, i.e., a continuous map $g: Y \to X$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$.

(a) Prove that $f: X \to Y$ is an embedding if $f$ is continuous and injective provided that $X$ is compact and $Y$ Hausdorff.

Solution: $f$ is a continuous bijection of $X$ with im $f$. Since $X$ is compact and $Y$ Hausdorff, the claim follows.

(b) Find an embedding $\mathbb{R}P^2 \to \mathbb{R}^4$.

Solution: Look at the map $\bar{f}: S^2 \to \mathbb{R}^4, (x, y, z) \mapsto (x^2, y^2, xz, yz)$. Clearly, there is a map $f: \mathbb{R}P^2 = S^2/\pm \to \mathbb{R}^4$ such that $\bar{f} = f \circ p$, where $p$ is the quotient map $S^2 \to \mathbb{R}P^2$. This map $f$ is injective if the map $\bar{f}$ is $2 \to 1$, which is immediate.

Hint: If you view $\mathbb{R}P^2 = S^2/\pm$, you need to write down a $2 \to 1$ map $S^2 \to \mathbb{R}^4$.

please hand up 18-19 Tuesday, 30/3/2010 in class
20. Write down formulas for homeomorphisms $h_i: X_i \to Y_i$ for the pairs of topological spaces given below:

(a) $X = \mathbb{R}/\mathbb{Z}$, $Y = S^1$.

Solution: $[x] \mapsto e^{2\pi ix}$

(b) $X = \mathbb{R}^n$, $Y = \mathbb{R}P^{n-1}$ where $A$ is the subset (homeomorphic to $\mathbb{R}P^{n-1}$) of all lines lying in the span of the first $n$ vectors of the standard basis of $\mathbb{R}^{n+1}$.

Solution: $x \mapsto \text{span}\{(x, 1)\}$

(c) $X = S^1 \times S^1$, $Y = \{ (x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1 \}$.

Solution: $(u, v) \mapsto ((2 + \Re u)\Re v, (2 + \Re v)\Im u, 3v)$

21. Prove that a topological space is connected if any two points lie in a connected subset.

Solution: If $a, b \in X$, $X$ a topological space and $U, V \subset X$ are open, $a \in U, b \in V$ and $U \cap V = \emptyset$, $U \cup V = X$. If $a, b \in Q \subset X$, then $a \in U \cap Q, b \in V \cap Q$ and both $U \cap Q, V \cap Q$ are open subsets of $Q$ wrt the subspace topology. We also have the disjoint union $(U \cap Q) \cup (V \cap Q) = Q$. Hence $Q$ can not be connected.

22. Let $X$ be a Hausdorff space and $A, B \subset X$ compact disjoint subsets. Prove that there are open $U, V \subset X$ with $A \subset U$, $B \subset V$, $U \cap V = \emptyset$

Solution: For each $a \in A, b \in B$ let $U_{a,b}$ and $V_{a,b}$ be open and such that $a \in U_{a,b}, b \in V_{a,b}, U_{a,b} \cap V_{a,b} = \emptyset$.

For each $b \in B$ we have

$$\bigcup_{a \in A} U_{a,b} \supset A$$

By compactness, there is a finite subset $F_b \subset A$ with

$$\bigcup_{a \in F_b} U_{a,b} \supset A$$

We also have

$$b \in V_b := \bigcap_{a \in F_b} V_{a,b} \subset X \text{ open, } V_b \cap \bigcup_{a \in F_b} U_{a,b} = \emptyset \text{ and }$$

$$B \subset \bigcup_{b \in B} V_b$$

By compactness of $B$ we find $F \subset B$ finite with

$$B \subset \bigcup_{b \in F} V_b$$

Now check that

$$B \subset V = \bigcup_{b \in F} V_b, A \subset U := \bigcap_{b \in F} \bigcup_{a \in F_b} U_{a,b}, U \cap V = \emptyset$$

23. Recall that a topological group is a triple $(G, \mu, T)$ so that $(G, \mu)$ is a group and $(G, T)$ a topological space so that the group multiplication $\mu: G \times G \to G$ and inversion $G, g \mapsto g^{-1}$ are continuous with respect to $T$ (and the product topology on $G \times G$). If $H \subset G$ is a subgroup, we endow the quotient $G/H = \{ gH \mid g \in G \} = G/\sim, g \sim g' \Leftrightarrow g^{-1}g' \in H$, with the quotient topology $\pi_* T$ where $\pi: G \to G/H, g \mapsto gH$ is the quotient map.

(a) Let $G = \mathbb{R}$ be the additive group of the real numbers and $H = \mathbb{Q}$. Is $\mathbb{R}/\mathbb{Q}$ with this topology Hausdorff, compact, connected?

Solution: Let $p: \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the quotient map. Assume $\emptyset \neq V \subset \mathbb{R}/\mathbb{Q}$ is open. Then $p^{-1}(V) \subset \mathbb{R}$ is open and $p^{-1}V \neq \emptyset$, say $x \in p^{-1}V$. We want to show that $p^{-1}(V) = \mathbb{R}$: Since $p^{-1}V \subset \mathbb{R}$ is open, there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset p^{-1}(V)$. Let $y \in \mathbb{R}$ be arbitrary. Then there is $q \in \mathbb{Q}$ such that $|x - y - q| < \epsilon$, hence $y + q \in (x - \epsilon, x + \epsilon) \subset p^{-1}(V), p(y + q) \in V$. But $p(y + q) = p(y)$ since $q \in \mathbb{Q}$, and we have shown that $y \in p^{-1}(V)$. This proves $p^{-1}(V) = \mathbb{R}$, hence $V = \mathbb{R}/\mathbb{Q}$. Thus any nonempty open set of $\mathbb{R}/\mathbb{Q}$ is all of $\mathbb{R}/\mathbb{Q}$, i.e. the quotient topology is the trivial one, $\{\emptyset, \mathbb{R}/\mathbb{Q}\}$. It is now immediate that this space is connected, compact, not Hausdorff.

(b) Prove that $O(n)/H$ is Hausdorff if $H \subset O(n)$ is closed.

(c) Let $H \subset O(n+1)$ be the subgroup (isomorphic to $O(n)$)

$$H := \left\{ \left( \begin{array}{c} 1 & 0 \\ 0 & A \end{array} \right) \mid A \in O(n) \right\}$$

and $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. Prove that the map

$$O(n+1)/H \to S^n$$

$$gH \mapsto ge_1$$

is a homeomorphism.
A scalar product on a real vector space $V$ is a symmetric positive definite bilinear form, i.e. a map

$$g: V \times V \to \mathbb{R}$$

such that for all $x, y, z \in V$, $\lambda \in \mathbb{R}$ we have

$$g(x, y) = g(y, x)$$
$$g(x + z, y) = g(x, y) + g(z, y)$$
$$g(\lambda x, y) = \lambda g(x, y)$$
$$g(x, x) = 0 \iff x = 0$$

Recall that if $g$ and $g'$ are scalar products, $\dim V < \infty$, then there is a Hermitian (wrt $g$) linear map $A: V \to V$ such that $g'(x, y) = g(x, Ay)$ for all $x, y \in V$. Any such linear map has a square root $\sqrt{A}$, i.e. the unique positive definite Hermitian linear map with $\sqrt{A}^2 = A \circ \sqrt{A} = A$.

We have a group action of $GL(n\mathbb{R})$ on the set $\mathcal{M}_n$ of all scalar products of $\mathbb{R}^n$ given by

$$GL(n, \mathbb{R}) \times \mathcal{M}_n = \mathcal{M}_n$$
$$(A, g) \mapsto A^*g, \text{ where}$$
$$A^*g(x, y) = g(Ax, Ay) \forall x, y \in \mathbb{R}^n$$

(a) Prove that this action is transitive.
(b) What is the isotropy group of this action at the standard scalar product.

**please hand up 20-24 Tuesday, 13/4/2010 in class**

25. Let $(a_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$ be a sequence in a set $X$. Recall that a sequence $(b_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$ is a subsequence of $(a_n)_{n \in \mathbb{N}}$ if there is an increasing map $\phi: \mathbb{N} \to \mathbb{N}$ (i.e. $\phi(n + 1) > \phi(n)$ for all $n \in \mathbb{N}$) such that for all $n \in \mathbb{N}$ we have $b_n = a_{\phi(n)}$. One says “$(a_n)_{n \in \mathbb{N}}$ contains the subsequence $(b_n)_{n \in \mathbb{N}}$”.

A topological space $X$ is called sequentially compact if any sequence in $X$ contains a convergent subsequence.

Prove that a compact metric space is sequentially compact.

**Hint:** Conversely a sequentially compact metric space is compact.

26. Let $X$ be a topological space and $A \subset X$. A **interior point** of $A$ is a point $p \in A$ such that there is $U \subset X$ open with $x \in U \subset A$, “$A$ contains an open neighbourhood of $x$”. The interior $A^\circ$ of $A$ is the set of interior points of $A$.

An accumulation point (cluster point, limit point) of $A$ is a point $x \in X$ such that for any open $U \subset X, x \in U$, we have that $A \cap U \setminus \{x\} \neq \emptyset$. The closure $\overline{A}$ of $A$ is the union of $A$ with the set of accumulation points of $A$.

The boundary of $A$ is $\partial A = \overline{A} \setminus A^\circ$.

Prove that for any subset $A$ of a topological space $(X, T)$, we always have

(a) $$A^\circ = \bigcup_{U \in T, U \subset A} U.$$  

(b) $$\overline{A} = \bigcap_{A \subset Q \subset X, Q \text{ closed}} Q = X \setminus (X \setminus A)^\circ \quad (3)$$

**Solution:** We trivially have

$$A \subset \bigcap_{A \subset Q \subset X, Q \text{ closed}} Q \subset X \setminus (X \setminus A)^\circ$$

Let $x \in X$ be an accumulation point of $A$ and $Q \subset X$ closed with $A \subset Q$. Assume $x \not\in Q, x \in X \setminus Q \subset T$. But by the definition of an accumulation point, there must be $a \in A \cap (X \setminus Q)$, hence we have $x \in Q$ which shows that

$$\overline{A} \subset \bigcap_{A \subset Q \subset X, Q \text{ closed}} Q \subset X \setminus (X \setminus A)^\circ \quad (4)$$

Now let $x \in X \setminus (X \setminus A)^\circ$, i.e. $x \not\in (X \setminus A)^\circ$ which means that

$$\not\exists U \in T : x \in U \subset X \setminus A$$

or

$$\forall U \in T, x \in U : U \cap A \neq \emptyset$$

which is equivalent to $x \in A$. This shows that $X \setminus (X \setminus A)^\circ \subset \overline{A}$. Together with (4) we get (3).

27. Let $\sim$ be the equivalence relation on $\mathbb{R}$ with $x \sim y \iff x - y \in \mathbb{Z}$. Endow $\mathbb{R}$ with the standard topology and consider $X = \mathbb{R}/\sim$ with the quotient topology. For the following subsets $A_i \subset X$ determine the closure $\overline{A_i}$, the interior $A^\circ$, the boundary $\partial A_i$ and state whether $A_i$ is connected, compact. Below, we denote by $|x| = x + \mathbb{Z} = \{y \in \mathbb{R} \mid y \sim x\}$ the equivalence class of $x \in \mathbb{R}$.
(a) \( A_a = \{ \lfloor 1/n \rfloor \mid n \in \mathbb{N} \} \)

(b) \( A_b = \{ 1/2, 1/3 \} \)

(c) \( A_c = \{ \lfloor n/2^k \rfloor \mid n, k \in \mathbb{N} \} \)

(d) \( A_d = \{ x \mid 0 < x \leq 1 \} \)

28. Consider the set \( X = \mathbb{R}^\mathbb{Q} \) with the product topology. Thus a set \( U \subset \mathbb{R}^\mathbb{Q} \) is open if for each \( f \in U \), \( f: \mathbb{Q} \to \mathbb{R} \), there is a finite set \( \{ q_1, \ldots, q_r \} \subset \mathbb{Q} \) and \( \epsilon \in \mathbb{R} \) such that \( \{ g: \mathbb{Q} \to \mathbb{R} \mid |g(q_i) - f(q_i)| < \epsilon \} \subset U \).

Hint: Prove this! For the following subsets \( A_i \subset X \) determine the closure \( \overline{A_i} \), the interior \( A^\circ_i \), the boundary \( \partial A_i \). Say whether \( A_i \) is connected or not and whether \( A_i \) is compact or not.

(a) \( A_a = \{ f: \mathbb{Q} \to \mathbb{R} \mid \forall n \in \mathbb{N} : f(n) < 1 \} \)

Solution: \( A_a^\circ = \emptyset, \overline{A_a} = \{ f: \mathbb{Q} \to \mathbb{R} \mid \forall n \in \mathbb{N} : f(n) \leq 1 \} \), \( \partial A_a = \overline{A_a} \). \( A_a \) is connected and not compact.

(b) \( A_b = \{ f: \mathbb{Q} \to \mathbb{R} \mid f(0) \in \mathbb{Z} \} \)

Solution: \( A_b^\circ = \emptyset, \overline{A_b} = A_b = \partial A_b \). \( A_b \) is not connected and not compact.

(c) \( A_c = \{ f: \mathbb{Q} \to \mathbb{R} \mid |f(0)| \leq 1 \text{ and } \forall x \in \mathbb{R} \setminus \{ 0 \} : f(x) = 0 \} \)

Solution: \( A_c^\circ = \emptyset, \overline{A_c} = \partial A_c = A_c \). \( A_c \) is connected and compact.

(d) \( A_d = \{ f: \mathbb{Q} \to \mathbb{R} \mid f(0), f(1), f(2) < 1 \} \)

Solution: \( A_d^\circ = A_d \), \( \overline{A_d} = \{ f: \mathbb{Q} \to \mathbb{R} \mid f(0), f(1), f(2) \leq 1 \} \), \( \partial A_d = \{ f: \mathbb{Q} \to \mathbb{R} \mid f(0) = 1 \text{ or } f(1) = 1 \text{ or } f(2) = 1 \} \).

\( A_d \) is connected and not compact.

please hand up 25-28 by Friday, 23/4/2010
If \(X,Y\) are topological spaces and \(A \subset X, B \subset Y\) subsets, we denote by
\[
f: (X,A) \to (Y,B)\quad \text{a map with } \ f(A) \subset B.
\]
Recall that the composition of two loops \(\omega, \mu: ([0,1], \{0,1\}) \to (X,x_0)\) in a topological space \((X,x_0)\) with base point is defined by
\[
\omega * \mu(t) = \begin{cases} 
\omega(2t) : 0 \leq t \leq 1/2 \\
\mu(2t-1) : 1/2 \leq t \leq 1 
\end{cases}
\]
Complete the proof of associativity of composition of homotopy classes of loops by constructing a continuous function
\[
\phi: [0,1] \to [0,1] \quad \text{with } \phi(0) = 0, \phi(1) = 1
\]
so that for all topological spaces \((X,x_0)\) with base point and all loops \(\omega, \mu, \sigma: ([0,1], \{0,1\}) \to (X,x_0)\) we have
\[
((\omega * \mu) * \sigma) \circ \phi = \omega * (\mu * \sigma)
\]
Let \(A \subset X\) be a subset of a topological space \(X\) and denote by \(i\) the inclusion, i.e. the map
\[
i: A \to X \\
a \mapsto a
\]
The subset \(A\) is a deformation retract of \(X\) if there is a continuous map \(r\), the “retraction”,
\[
r: X \to A \quad \text{with } \ r|_A = \text{id}_A
\]
and so that
\[
i \circ r \simeq \text{id}_X \text{ rel } A. \tag{5}
\]
(Sometimes, the homotopy \(H\) in (5) is called the deformation retraction.) Thus there is a continuous map
\[
H: X \times [0,1] \to X
\]
such that
\[
\forall x \in X, a \in A, t \in [0,1]: H(x,1) = x, r(x) = H(x,0) \in A, H(a,t) = a.
\]
Which of the following subsets \(A \subset \mathbb{R}^2\) are deformation retracts of \(\mathbb{R}^2\)?
(a) \(A = \{(0,0)\}\)
(b) \(A = \mathbb{R} \times \{0\}\)
(c) \(A = \{(x,y) | x, y \in \mathbb{R}, x \geq y\}\)
(d) \(A = \{(x,y) | x, y \in \mathbb{R}, x < 0\}\)
A continuous map \(f: X \to Y\) of topological spaces \(X, Y\) is a homotopy equivalence, if
\[
\exists g: Y \to X \text{ continuous } : f \circ g \simeq \text{id}_Y \text{ and } g \circ f \simeq \text{id}_X.
\]
Prove that the inclusion of a deformation retract is a homotopy equivalence.

Please hand up 29-31 Tuesday, 4/5/2010 in class
Some problems for the study week

32. This problem is for those who feel uncomfortable about formalism. Please write down the definitions of the following notions in two ways: firstly as a formula, secondly as precisely as possible in words, using as few as possible mathematical symbols. Use the following example as a guideline:

notion to be defined: Topology on a Set

\[ T \text{ is a topology on the set } X \quad \overset{\text{def}}{\iff} \quad \begin{cases} \emptyset, X \in T \\ Z \subset T \quad \Rightarrow \quad \bigcup_{W \in Z} W \in T \\ Z \subset T, \#Z < \infty \quad \Rightarrow \quad \bigcap_{W \in Z} W \in T \end{cases} \]

now in plain english:
A topology on a set \( X \) is a family of subsets of \( X \) with the following three properties:
The empty set and \( X \) are elements of this family,
any union of elements of the family is an element of the family,
the intersection of finitely many elements of the family is an element of the family.

Write down definitions for: closed subset of a topological space, connected, compact, Hausdorff topological space, metric topology, quotient topology, product topology, basis for a topology, topology generated by a basis, closure, interior, boundary of a subset of a topological space, homotopy, homotopy equivalence, deformation retract.

Make sure you write complete sentences. Be precise! For instance, carefully distinguish between “\( a \) is element of \( A \)” and “\( a \) is a subset of \( A \)”.

33. Is the open ball \( B^n(0) = \{ x \in \mathbb{R}^n \mid \|x\| < 1 \} \) a deformation retract of \( \mathbb{R}^n \)?

34. A topological space \( X \) is contractible if there is \( a \in X \) such that \( \{a\} \) is a deformation retract. The cone over a topological space \( X \) is the quotient \( CX = X \times [0,1]/\sim \) where the equivalence relation \( \sim \) is given by

\( (x,1) \sim (x',1) \quad \text{for all} \quad x,x' \in X. \)

Prove that the cone of any topological space is contractible.

**Hint:** Prove that \( \{X \times \{1\}\} \subset CX \) is a deformation retract.

35. Let \( X = S^1 \times [0,1]/\sim \) (with the quotient topology) where the equivalence relation \( \sim \) is given by

\( (z,0) \sim (z^6,1) \quad \text{for} \quad z \in S^1. \)

Find a presentation for the fundamental group of \( X \).

**Solution:** Look at the following square

```
   a  b  b  b
   b  b  b  b
   b

   a
```

The space \( X \) is obtained by gluing a square as indicated in the picture. Thus the the four corners and the five points on the right (the dots) are all identified to one point (the base point) and the two segments on top and bottom (denoted by \( a \)) and the one segment \( b \) on the left and the six segments \( b \) on the right are identified as indicated by the arrows. Thus the fundamental group of the whole space is generated by two loops arising from \( a \) and \( b \). The loop \( r = ab^{-6}a^{-1}b \) is once around the whole square and can be contracted along the line to the upper left corner to the base point. Thus the fundamental group of \( X \) is

\( \langle a,b \mid ab^{-6}a^{-1}b \rangle. \)

36. Let \( X,Y \) be sets and \( f: X \to Y \) be a map.

(a) Let \( B \) be a basis for a topology of \( Y \) and consider the family

\[ f^{-1}(B) := \{ f^{-1}(B) \mid B \in \mathcal{B} \} \]

of subsets of \( X \). Prove that \( f^{-1}(\mathcal{B}) \) is a basis for a topology of \( X \).

**Hint:** \( f^{-1}(\mathcal{B}) = \{ x \in X \mid f(x) \in \mathcal{B} \} \)
(b) Show (by a counterexample) that if $B$ is a basis for a topology on $X$, then $f(B) := \{f(B) \mid B \in B\}$ need not be a basis for a topology on $Y$. What happens if $f$ is surjective?

37. Let $X, Y$ be path connected topological spaces and assume $f : X \to Y$ is a homeomorphism. Prove that $X$ is simply connected if and only if $Y$ is simply connected.

Hint: A topological space $X$ is simply connected if $X$ is path connected and any loop $f : [0,1] \to X$ is homotopic (rel$\{0,1\}$) to the constant loop at $f(0) = f(1)$.

Solution: If $\omega : [0,1] \to Y$ is a loop and $X$ is simply connected, then there is a homotopy $H : [0,1] \times [0,1] \to X$ of loops $f^{-1} \circ \omega \simeq_H c_{f^{-1}(\omega(0))}$, where $c_x$ denotes the constant loop at $x$. Then the composition $f \circ H$ gives a homotopy $\omega \simeq_{f \circ H} c_{\omega(0)}$. For the converse, apply the same argument with $X, Y$ interchanged.

38. Let $A \subset \mathbb{R}^n$, $n > 1$, be a nonempty connected open subset, and $x_1, \ldots, x_r \in A$ be finitely many points. Prove that $A \setminus \{x_1, \ldots, x_r\}$ is connected.
39. Consider the following topologies on \( \mathbb{R} \):

(a) \( T_a = \{ (-\infty, a) \mid a \in \mathbb{R} \} \cup \{ \emptyset, \mathbb{R} \} \),

(b) \( T_b = \{ U \in \mathcal{P}(\mathbb{R}) \mid \mathbb{R} \setminus U \text{ finite} \} \).

For each of the topological spaces \( (\mathbb{R}, T_i) \), \( i = a, b \), decide whether it is compact, Hausdorff, connected. Also say whether the topology \( T_i \) is induced by a metric on \( \mathbb{R} \).

40. Let \( X \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \) be the double cone

\[
X = \{ (z, h) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 = h^2 \leq 1 \}
\]

and let \( \sim \) be the equivalence relation on \( X \) generated by

\[
(z, 1) \sim (\epsilon z, 1) \text{ and } (z, -1) \sim (\epsilon^2 z, -1)
\]

where

\[
\epsilon = e^{2\pi i / 12}
\]

is a primitive 12th root of unity. Compute (i.e. find a presentation for) the fundamental group of the quotient \( X/\sim \).

41. On the set \( X = C([0, 1], \mathbb{R}) \) of all continuous functions \([0, 1] \to \mathbb{R}\) consider the metric

\[
d: X \times X \to \mathbb{R}_0^+ \quad , \quad d(f, g) = \int_0^1 |f - g| \quad \text{for all } f, g \in X .
\]

For each of the following subsets \( A_i \subset X \) find the closure \( \bar{A_i} \) and the interior \( A_i^o \) of \( A_i \) with respect to the metric topology \( T(d) \) on \( X \). Furthermore state whether or not \( A_i \) is connected, whether or not \( A_i \) is compact, and whether or not \( A_i \) is a deformation retract of \( X \).

(a) \( A_a = \{ f \in X \mid \forall t \in [0, 1] : |f(t)| < 1 \} \),

(b) \( A_b = \{ f \in X \mid \forall n \in \mathbb{N} : 1/f(1/n) \in \mathbb{N} \} \),

42. Let \( X = (\{0, 1\}, \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}) \) (i.e. a topological space with two points and the discrete topology) and \( A \) be a set.

(a) Prove that \( X^A \) is compact wrt the product topology.

(b) Prove that a topological space \( Y \) is disconnected if there is a surjective continuous map \( f: Y \to X \).