The Computation of η-Invariants on Manifolds with Free Circle Action

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ABSTRACT

We present an explicit procedure to compute η-invariants. This also yields a topological formula for adiabatic limits and simplifies the calculation of Kreck-Stolz-invariants detecting components of the space of positive scalar curvature metrics.

1 Introduction

Let M be a Riemannian Spin-manifold of positive scalar curvature carrying a free and isometric action of the circle S^1 with geodesic orbits. We compute the η-invariant of twisted Dirac-operators on M. We list as an example the explicit result for the (generalized) Berger spheres of dimension ≤ 11 (i.e. the odd dimensional spheres with a metric obtained by rescaling the standard metric in direction of the orbits of the circle action given by complex multiplication). As a second application we derive a formula for the adiabatic limit of η-invariants.

The η-invariant of such an operator D is an analytic regularization of the asymmetry of the spectrum of D. It is obtained by evaluating the meromorphic extension of the Dirichlet series $\sum |\lambda|^{-s}\mathrm{sign}\,(\lambda)$, which converges for large $\mathrm{Re}\, s$, at $s = 0$ (cf. [10], [3]). In the Atiyah-Patodi-Singer index formula [1] for manifolds W with boundary M it arises as the contribution of M to the index of D on W.

In fact our calculation will be based on the index theorem. We prove a vanishing theorem for the index of Dirac operators on disc bundles. To this end we construct a Riemannian metric g_{DE} and a connection $\nabla^{\alpha_{DE}}$ on the canonical bundle of an appropriate Spin^c-structure α_{DE} on the disc bundle DE associated to M, such that the scalar curvature of g_{DE} exceeds the absolute value of the smallest eigenvalue of the curvature endomorphism of $\nabla^{\alpha_{DE}}$. By the vanishing theorem of Hitchin-Lichnerowicz for the kernel of Dirac-operators the index of the Dirac-operator on DE is trivial. It then follows from the Atiyah-Patodi-Singer index formula that the η-invariant of M is given by twice the integral over DE of the $\hat{A}$-form twisted with the canonical line bundle of the Spin^c-structure on DE.

The last section contains the formulae for the curvature form on DE, followed by a recipe to compute this integral. As a direct application of the vanishing theorem we finally derive a formula for the limit of η-invariants of circle bundles when the orbits of the circle action are shrinked.

In some cases the η-invariant of the Dirac operator has been computed directly out of the Dirac spectrum, e.g. by Hitchin [11] for the 3-dimensional Berger spheres, by Seade-Steer [17] for quotients of $PSL_2(\mathbb{R})$ by Fuchsian groups. Furthermore there are general formulae by Bismut-Cheeger [7] and Dai [8] for the adiabatic limit of the η-invariant in fibrations. This has been made explicit for S^1-bundles by W. Zhang in [18] thus also deriving the formula for the adiabatic limit of the η-invariant. The vanishing theorem 2.2 gives a straightforward computation of the invariants used by Kreck and Stolz in [13] to find manifolds with a nonconnected space of positive sectional curvature metrics (see also [9]). In contrast to [9] our approach in computing these invariants avoids the roundabout through η-invariants by giving an explicit geometric construction.

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2 A Vanishing Theorem for Disc Bundles

By (M, g) we will always denote a Riemannian manifold M of odd dimension carrying a free isometric action of the circle S^1 with geodesic orbits, an equivariant Spin^c-structure α, an equivariant Hermitian vector bundle ζ and unitary connections ∇^α and ∇^ζ on the canonical line bundle $\xi(\alpha)$ of α and on ζ respectively. We assume that the orbits of the circle action on the vector bundles $\xi(\alpha)$ and ζ are parallel with respect to these connections. Furthermore let s_M denote the scalar curvature of M and let R^α and R^ζ be the curvature tensors of ∇^α and ∇^ζ respectively. Let D_ζ be the Spin^c-Dirac operator twisted with the connection ∇^ζ. We follow the conventions in [14].

The orbit space $B = M/S^1$ then is a manifold and there is a metric g_B on B such that the quotient map $\pi : M \to B$ becomes a principal S^1-bundle and a Riemannian submersion with totally geodesic fibres. The Spin^c-structure α, the vector bundle ζ and the connections are induced from a Spin^c-structure α_B, a vector bundle ζ_B and connections over B.

For a 2-form $\mu \in \Omega^2(X; \mathfrak{u}(S \otimes \zeta))$ with values in the skew-Hermitian endomorphisms of the twisted spinor bundle $S \otimes \zeta$ over a manifold X we define $||\mu|| \in C^\infty(X)$ by

$$||\mu||(x) := -\min\{\langle \mathcal{E}(\mu)s \mid s\rangle \mid s \in (S \otimes \zeta)_x, ||s|| = 1\} ,$$

where the Hermitian endomorphism $\mathcal{E}(\mu)$ of $S \otimes \zeta$ is given as

$$\mathcal{E}(\mu)(\sigma \otimes \varepsilon) = \frac{1}{2} \sum_{j,k} \mu(e_j, e_k)(e_j e_k \sigma \otimes \varepsilon)$$

for $\sigma \in S_x$, $\varepsilon \in \zeta_x$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x X$. For 2-forms $\mu, \nu \in \Omega^2(X; \mathfrak{u}(S \otimes \zeta))$ we have the triangle inequality $||\mu|| + ||\nu|| \geq ||\mu + \nu||$. For $x \in M$ let $\{e_1, \ldots, e_n\}$ be such that $R^\alpha = \sum_{i \leq (n-1)/2} \lambda_i e_i \wedge e_{[i+(n-1)/2]}$.

Then (see [11], [14])

$$||R^\alpha \otimes 1|| = \sum_{i\leq (n-1)/2} |\lambda_i| \ . \tag{2.1}$$

On the disc bundle DE of the associated complex line bundle $E = M \times_{S^1} \mathbb{C}$ we then have an equivariant vector bundle ζ_{DE} and an equivariant Spin^c-structure α_{DE} both extending the corresponding data on $M = \partial DE$. We consider the Spin^c-structure induced from α_B and the Spin$^c(2)$-structure on the vector bundle E. Assume that g_{DE} is a Riemannian metric on the manifold DE and $\nabla^{\zeta_{DE}}$ and $\nabla^{\alpha_{DE}}$ are connections on ζ_{DE} and on $\xi(\alpha_{DE})$ such that in a suitable collar neighbourhood U of M they are induced from a product structure $U \cong M\times]-\epsilon, 0]$. In this setting there is defined a Spin^c-Dirac operator $D_{\zeta_{DE}}$ acting on twisted spinors satisfying the Atiyah-Patodi-Singer boundary conditions and whose tangential operator is D_ζ, cf [1]. For the index of the operator $D_{\zeta_{DE}}$ we have the following vanishing theorem:

Theorem 2.2 *If*

$$\frac{1}{4}s_M(x) > ||\frac{1}{2}R^\alpha \otimes 1 + 1 \otimes R^\zeta||(x) \ \textit{for all } x \in M$$

then the index of the Dirac operator $D_{\zeta_{DE}}$ vanishes.

The Atiyah-Patodi-Singer Index Theorem [1] then gives

Corollary 2.3

$$\eta(D_\zeta) = 2\int_{DE} \mathrm{ch}\,(\nabla^{\zeta_{DE}}) \wedge e^{c_1(\nabla^{\alpha_{DE}})/2} \wedge \hat{A}(p(g_{DE})) \ . \tag{2.4}$$

The last section contains a recipe to explicitly compute the η-invariant of certain homogeneous spaces, by calculating the integral in (2.4).

If ζ is the 0-bundle, M carries a Spin-structure and α is its complexification, then $R^\alpha = 0$ and the condition is just $s_M > 0$. The corresponding Dirac operator has the same spectrum as the Dirac operator of the Spin-structure.

The Bochner formula for twisted Dirac operators (see [11], [15], [14]) states that the Dirac-Laplacian satisfies the formula

$$D_\zeta^2 := \nabla^*\nabla + \frac{1}{4}s + \mathcal{E}(\frac{1}{2}R^\alpha \otimes 1 + 1 \otimes R^\zeta) \ . \tag{2.5}$$

The condition of the theorem thus is positivity of the order 0 term in this formula on M.

Proof of Theorem 2.2: Since the index does not depend on the metric nor on the curvature form in the interior of DE it suffices to construct a metric g_{DE} on DE and a connection $\nabla^{\alpha_{DE}}$ on $\xi(\alpha_{DE})$ extending the metric g and the connection ∇^α on M, such that the estimate $\frac{1}{4}s_{DE} > ||\frac{1}{2}R^{\alpha_{DE}} \otimes 1 + 1 \otimes R^{\zeta_{DE}}||$ holds, where s_{DE} is the scalar curvature of g_{DE} and $R^{\alpha_{DE}}$ and $R^{\zeta_{DE}}$ are the curvature tensors of $\nabla^{\alpha_{DE}}$ and $\nabla^{\zeta_{DE}}$. In [2] before Theorem 3.9 it is shown, that the usual Lichnerowicz argument then also shows, that the index of the Dirac-operator vanishes.

Since the fibres of M are assumed totally geodesic, they are all isometric to a circle $S^1_\rho \hookrightarrow \mathbb{C}$ of radius ρ. For some $\delta \in \mathbb{R}^+$ to be determined later the disc $D^2 \subset \mathbb{C}$ of radius δ will be endowed with a metric such that ∂D^2 is isometric to S^1_ρ. The map $M \times [0,\delta] \to DE = M \times_{S^1} D^2$ is a Diffeomorphism when restricted to $M \times]0,\delta]$. Let $\tilde{u}$ be the fundamental vector field of the S^1-action, $u := \tilde{u}/|u|$, and v be the radial derivative i.e. the derivative with respect to the interval-factor. Let g^τ be the canonical variation of M i.e. the family of metrics on M defined by rescaling the orbits of the circle action: $g^\tau(x,y) := g(x,y) + (\tau^2 - 1)g(x,\tilde{u})g(y,\tilde{u})$. For an odd smooth function f on $\mathbb{R}$ with $f'(0) = 1$ mapping $[0,\delta] \to [0,\rho]$ and constant ρ on $[\gamma,\delta]$ the metric $g^{f(\tau)} \times d\tau^2$ on $M \times]0,\delta] \hookrightarrow DE$ extends to give a metric g_{DE} on all of DE such that $\pi : DE \to B$ is a Riemannian submersion with totally geodesic fibres and DE carries a product metric near its boundary M.

Since v commutes with all basic vectorfields of DE we get

$$\nabla_v v = \nabla_v u = 0,\ \nabla_u v = \frac{f'}{f}u,\ \nabla_u u = -\frac{f'}{f}v, \text{ and } \nabla_a v = 0, \tag{2.6}$$

for every horizontal vector field a. We will need the scalar curvature of g_{DE}. By O'Neill's formulae (see [6]) this is given by

$$s_{DE} = s_F + s_B - ||A||^2 \ ,$$

where $A_x y = \mathcal{V}\nabla_{\mathcal{H}x}\mathcal{H}y = \frac{1}{2}\mathcal{V}[\bar{x},\bar{y}]$, $\mathcal{V}$ and $\mathcal{H}$ denoting the vertical and horizontal projections of the Riemannian submersion respectively. Let $e = (m,\tau) \in DE$. From (2.6) we get the scalar curvature of the fibre $F = D^2$ of the submersion $DE \to B$ as

$$s_F(e) = -2\frac{f''(\tau)}{f(\tau)}.$$

Furthermore we compute

$$\begin{aligned} ||A||^2(e) &= \sum_{i,j} ||A_{\bar{h}_i(e)}\bar{h}_j(e)||^2 = \sum_{i,j}\frac{1}{4}||\mathcal{V}[\bar{h}_i,\bar{h}_j](e)||^2 \\ &= \sum_{i,j}\frac{1}{4}f(\tau)^2||\mathcal{V}[\bar{h}_i,\bar{h}_j](m)||^2 = f(\tau)^2||A||^2(m), \end{aligned}$$

because the map $M \to DE$, $m = (m,\delta) \mapsto (m,\tau)$ preserves the vectorfields $\bar{h}_i$ and maps u to $f(\tau)u$. The scalar curvature of DE is thus estimated by

$$\begin{aligned} s_{DE}(e) &= s_F(e) + s_B(\pi(m)) - ||A||^2(e) \\ &= -2\frac{f''(\tau)}{f(\tau)} + s_M(m) + (\rho^2 - f(\tau)^2)||A||^2(m) \qquad (2.7) \\ &\geq -2\frac{f''(\tau)}{f(\tau)} + s_M(m). \end{aligned}$$

The equivariant Spin^c-structure α_M bounds the Spin^c-structure α_{DE} on the disc bundle DE. The canonical bundle of α_{DE} is $\xi(\alpha_{DE}) = \pi^*(\xi(\alpha_B) \otimes E)$. Now we proceed to suitably extend the connection ∇^α. By equivariance we find a connection ∇^{α_B} on $\xi(\alpha_B)$ such that $\nabla^\alpha = \pi^*\nabla^{\alpha_B}$. Denote by ∇^E the connection on E induced from the Riemannian metric on M and let ∇^0 be the flat connection on the pull back of E over $DE \setminus B$ induced from its canonical trivialization. Now pick a smooth decreasing function

$$\psi : \mathbb{R}_0^+ \longrightarrow [0,1]$$

which is constant 0 in the intervall $[\gamma, \delta]$ and 1 in $[0, \alpha]$ for suitable $\alpha \in]0, \gamma[$. The function obtained by composing with the distance $d(\cdot, B)$ from the 0-section is also denoted by ψ.

Define a connection on $\xi(\alpha_{DE})$ by

$$\nabla^{\alpha_{DE}} = (\pi^*\nabla^{\alpha_B} \otimes 1 + 1 \otimes (\psi\pi^*\nabla^E + (1-\psi)\nabla^0).$$

The curvature tensor of this connection is

$$\begin{aligned} R^{\alpha_{DE}} &= \pi^* R^{\alpha_B} + d\psi \wedge (\pi^*\nabla^E - \nabla^0) + \psi\pi^* R^E \\ &= \pi^* R^{\alpha_B} - i\frac{\psi'}{f} u \wedge v + \psi\pi^* R^E, \end{aligned} \tag{2.8}$$

where we have written u and v for the 1-forms $u = \langle u \mid \cdot \rangle$, $v = \langle v \mid \cdot \rangle$. In view of (2.7) we search functions f and ψ such that for every $e = (m, \tau) \in DE$ we have

$$4\,||\frac{1}{2}R^{\alpha_{DE}} \otimes 1 + 1 \otimes R^{\zeta_{DE}}||(e) \le -2\frac{f''(\tau)}{f(\tau)} + s_M(m).$$

By the triangular inequality for $||\cdot||$ we estimate using (2.1) and substituting $R^\alpha = \pi^* R^{\alpha_B}$:

$$||\frac{1}{2}R^{\alpha_{DE}} \otimes 1 + 1 \otimes R^{\zeta_{DE}}|| \le ||\frac{1}{2}(\pi^* R^{\alpha_B} \otimes 1 + 1 \otimes R^{\zeta_{DE}}|| - \frac{\psi'}{2f} + \frac{\psi}{2}||\pi^* R^E \otimes 1||.$$

By the assumption of the theorem

$$s := \frac{1}{2}\min(s_M - 4\,||\frac{1}{2}R^\alpha \otimes 1 + 1 \otimes R^\zeta||)$$

is positive. Let m be a real number with $m > s/2$ and $m > ||R^E \otimes 1||(b)$ for all $b \in B$. Then the theorem is proved if we can solve the differential estimate

$$-\frac{f''}{f} + s \ge 2\left(-\frac{\psi'}{2f} + \frac{\psi}{2}m\right) = -\frac{\psi'}{f} + \psi m \tag{2.9}$$

for functions f and ψ as before.

For every ϱ, β with $0 < \varrho < \rho$ and $0 < \beta < \varrho\pi/2$ let δ be a real number and $f : \mathbb{R}_0^+ \longrightarrow [0, \rho]$ be function such that

$$\begin{aligned} f(r) &= \varrho \sin(r/\varrho), \text{ if } r \in [0, \beta], \\ f''(r) &\le 0 \text{ for all } r, \\ f(r) &\ge \varrho, \text{ if } r \ge \varrho\pi/2, \\ f(r) &\equiv \rho \text{ near } \delta, \text{ i.e for some } \gamma < \delta \text{ we have } f \equiv \rho \text{ on } [\gamma, \delta]. \end{aligned}$$

We will use the the following obvious fact about smooth functions:

Lemma 2.10 *Let F be a smooth real function such that $F' \geq 0$ and let $b > a$, $\Psi_b > \Psi_a > 0$ be real numbers with $F(b) - F(a) > \Psi_b - \Psi_a > 0$. Then there is a smooth real function Ψ which is constant near a and near b with $\Psi(b) = \Psi_b$, $\Psi(a) = \Psi_a$ and $0 \leq \Psi' \leq F'$*

We will show that one can find $\varrho \in]0, \rho]$, $\beta \in]0, \varrho\pi/2[$ and $\alpha \in]0, \beta[$ and a function $\psi : \mathbb{R}_0^+ \longrightarrow [0,1]$ with $\psi \equiv 1$ on $[0, \alpha]$ and $\psi \equiv 0$ near δ such that (f, ψ) solve (2.9). We get a solution of (2.9) on $[0, \alpha]$ if $-f''/f = 1/\varrho^2 > 2m$, so we impose the condition

$$2m\varrho^2 < 1. \tag{2.11}$$

Clearly

$$0 \leq -\psi' \leq -f'' - mf \tag{2.12}$$

implies (2.9) on $[0, \beta]$. By 2.10 we can extend ψ to $[0, \beta]$ such that ψ is constant near β, $\psi(\beta) < s/m$ and 2.12 holds if the condition

$$1 - \frac{s}{m} < \int_\alpha^\beta -f'' - mf = (1 - m\varrho^2)(\cos(\alpha/\varrho) - \cos(\beta/\varrho)) \tag{2.13}$$

is fulfilled. If we set $\psi \equiv \psi(\beta)$ on $[\beta, \varrho\pi/2]$ then (f, ψ) solve 2.9 on $[0, \varrho\pi/2]$. In order to get a solution on $[0, \delta]$ with $\psi \equiv 0$ near δ for some δ we solve $s \geq -\psi'/\varrho + m\psi(\beta)$ on $[\varrho\pi/2, \infty[$ for some extension of ψ which is constant near $\varrho\pi/2$ and δ. Again applying 2.10 we need to find δ such that

$$\int_{\varrho\pi/2}^\delta s - m\psi(\varrho\pi/2) \geq (s - m\psi(\varrho\pi/2))(\delta - \varrho\pi/2) > \psi(\varrho\pi/2) \tag{2.14}$$

holds.

Now choose ϱ sufficiently small such that $1 - s/m < 1 - m\varrho^2$. Then condition (2.11) holds and we can accomplish (2.13) by choosing α sufficiently close to 0 and β close to $\varrho\pi/2$. The values of $\psi(\varrho\pi/2) < s/m$ and ϱ now being fixed we can take δ sufficiently large to ensure that (2.14) holds.

3 Computation of the Eta-Invariant

In this section we will show how to compute the integral in Corollary 2.3. For simplicity we will confine ourselves to the untwisted Spin-case. The integral does not depend on the choice of the connection $\nabla^{\alpha_{DE}}$ on π^*E in the interior of DE, so we may take the vertical projection of the Levi-Civita-connection of the Riemannian metric for $\nabla^{\alpha_{DE}}$, identifying the bundle along the fibres of DE with π^*E. This corresponds to choosing $\psi = f'$ in the previous proof.

For the Riemannian curvature tensor on DE at a point $e = (m, \tau)$ we get from O'Neill's formulae [6], as before using the rescaling property of the map $(m, \delta) \mapsto (m, \tau)$:

$$\begin{aligned} \langle R^{DE}_{u,v} u \mid v \rangle &= \frac{f''(\tau)}{f(\tau)} \ , \\ \langle R^{DE}_{a,u} u \mid v \rangle &= 0 \ , \end{aligned}$$

$$
\begin{aligned}
\langle R^{DE}_{a,v} u \mid v\rangle &= 0 \ , \\
\langle R^{DE}_{a,v} b \mid v\rangle &= 0 \ , \\
\langle R^{DE}_{a,b} u \mid v\rangle &= \langle \nabla_{[a,b]} v \mid u\rangle = 2\ f'(\tau)\ \alpha(a,b) \ , \\
\langle R^{DE}_{a,b} c \mid v\rangle &= 0 \ , \\
\langle R^{DE}_{a,u} u \mid a\rangle &= \langle A_a u \mid A_a u\rangle(e) = f(r)^2\ \langle A_a u \mid A_a u\rangle^M(m) \ , \\
\langle R^{DE}_{a,v} u \mid b\rangle &= \langle \nabla_v \nabla_a b \mid u\rangle(e) = v\langle \nabla_a b \mid u\rangle(e) = f'(\tau)\ \alpha(a,b) \ , \\
\langle R^{DE}_{a,b} u \mid c\rangle &= f(r)\ \langle R^M_{a,b} u \mid c\rangle^M(m) \ , \\
\langle R^{DE}_{a,b} c \mid h\rangle &= \langle R^M_{a,b} c \mid h\rangle^M(m) \\
&\quad + (f(\tau)^2 - 1)(2\ \alpha(a,b)\ \alpha(c,h) - \alpha(a,h)\ \alpha(b,c) + \alpha(a,c)\ \alpha(b,h))
\end{aligned}
$$

where a, b, c and h denote horizontal vectors in $T_e DE$ and the corresponding vectors in $T_m M$ as well, and $\alpha(a,b) := \frac{1}{\rho}\langle A_a b \mid u\rangle(m) = \frac{1}{2\rho}\langle [a,b] \mid u\rangle(m) =.$ Writing x for the 1-form $\langle x \mid \cdot\rangle$, $x \in TDE$, and a^* for the 1-form $\alpha(a,\cdot)$, we express the curvature 2-form as

$$
\begin{aligned}
\langle R_{\cdot,\cdot} u \mid v\rangle &= \frac{f''(r)}{f(r)}\ u \wedge v + 2f'(r)\ \alpha \ , & (3.1)\\
\langle R_{\cdot,\cdot} u \mid a\rangle &= f(r)\ \langle R^M_{\mathcal{H}\cdot,\mathcal{H}\cdot} u \mid a\rangle^M - f(r)^2\ u \wedge \langle A_a u \mid A_{\mathcal{H}\cdot} u\rangle & \\
&\quad + f'(r)\ v \wedge a^* \ , & (3.2)\\
\langle R_{\cdot,\cdot} v \mid a\rangle &= -f'(r)\ u \wedge a^* \ , & (3.3)\\
\langle R_{\cdot,\cdot} a \mid b\rangle &= \langle R^M_{\mathcal{H}\cdot,\mathcal{H}\cdot} a \mid b\rangle^M + f(r)\langle R^M_{\mathcal{H}\cdot,u} a \mid b\rangle^M \wedge u & \\
&\quad + 2\ f'(r)\ \alpha(a,b)\ u \wedge v + (f(r)^2 - 1)\ (2\alpha(a,b)\ \alpha + a^* \wedge b^*) & (3.4)\\
&= \langle R^B_{\mathcal{H}\cdot,\mathcal{H}\cdot} a \mid b\rangle^B + f(r)\langle R^M_{\mathcal{H}\cdot,u} a \mid b\rangle^M \wedge u & \\
&\quad + 2\ f'(r)\ \alpha(a,b)\ u \wedge v + f(r)^2\ (2\alpha(a,b)\ \alpha + a^* \wedge b^*) \ . & (3.5)
\end{aligned}
$$

Recall that the $\hat{A}$-form is given by (see [5])

$$
\hat{A} = \det{}^{1/2} \frac{R/4\pi}{\sinh(R/4\pi)} \ .
$$

The first Chern-form of the bundle along the fibres of DE is obtained from (2.8), substituting $\psi = f'$ and $R^E = i\alpha$:

$$
c_1(\nabla^{\alpha_{DE}}) := \frac{1}{2\pi i}\pi^* R^{\alpha_B} + \frac{1}{2\pi}\left(\frac{f''(\tau)}{f(\tau)}\ u \wedge v + 2f'(\tau)\ \pi^*\alpha\right) \ . \qquad (3.6)
$$

Now assume $M = G/H$ homogeneous and that the circle action commutes with the action of G. Using the above formulae we express the characteristic form $e^{c_1(\nabla^{\alpha_{DE}})/2}\hat{A}(p(g_{DE}))$ at a point $e = (m,\tau) \in DE$ as

$$
e^{c_1(\nabla^{\alpha_{DE}})/2}\hat{A}(p(g_{DE}))(e) = P(f(\tau), f'(\tau), \frac{f''(\tau)}{f(\tau)})\,\mathrm{vol}\,(DE, g_{DE})
$$

with some polynomial P whose coefficients can be computed from (3.1) to (3.4). The volume form $\mathrm{vol}\,(DE, g_{DE})$ at e is $\mathrm{vol}\,(DE, g_{DE}) = f(\tau)\ \mathrm{vol}\,(M) \wedge v$ and we finally get

$$\begin{aligned}\frac{1}{2}\eta(M) &= \int_{DE} e^{c_1(\nabla^{\alpha_{DE}})/2}\hat{A}(p(g_{DE})) \\ &= \operatorname{vol}(M)\int_0^\delta P(f(\tau), f'(\tau), \frac{f''(\tau)}{f(\tau)})\ f(\tau)\,d\tau\end{aligned}$$

Now this integral can be calculated for a suitable function f.

Remark: Since the integral does not depend on the specific choice of f the integrand $f\ P(f, f', f''/f)$ is of the form

$$f\ P(f, f', f''/f) = \sum_{i,j} a_{i,j} f^i (f')^j + f'' \sum_{i,j} b_{i,j} f^i (f')^j = \left(\sum_{i,j} c_{i,j} f^i (f')^j\right)' ,$$

hence $a_{i-1,j+1}/i = b_{i,j-1}/j := c_{i,j}$ for $i, j > 0$. The value of the integral then is $\sum_i c_{i,0}\rho^i + \sum_j c_{0,j}$.

The sum $\sum_j c_{0,j}$ does not depend on ρ and will be computed in the next section on adiabatic limits from a topological formula.

The $c_{i,0}$ can be determined from the $a_{i,1}$ only. In order to compute the $a_{i,1}$ we may replace the expressions on the right hand side in (3.1), (3.3) and in (3.6) by 0, because in (3.1) to (3.4) the form v always occurs with a factor f'.

Thus terms involving f' and not v contribute to the constant part (i.e. independent of ρ) only. If M also carries a Spin-structure which is not equivariant but bounds a Spin-structure of DE then the conclusion of the vanishing theorem holds if the scalar curvature of M is positive because we only need to endow DE with a metric of positive scalar curvature to ensure that the index of the Dirac operator on DE vanishes. But this can be achieved as in the proof of that theorem. In order to compute the η-invariant we have to compute the integral over the $\hat{A}$-form only, but by the discussion above, this differs only by the term of order 0 in ρ from the equivariant case.

Example: The (generalized) Berger spheres M_ρ are obtained from the round sphere $M_1 = S^{n+1} \subset \mathbb{C}^{l+1}$ of odd dimension $n+1 = 2l+1$ of curvature 1 by shrinking the orbits of the S^1-action induced from complex multiplication. In this case we have $\langle R_{\mathcal{H}\cdot,\mathcal{H}\cdot} u \mid a\rangle^M = 0$ and $\langle R_{\mathcal{H}\cdot,\mathcal{H}\cdot} a \mid b\rangle^M = b \wedge a$. The horizontal distribution in TM has a complex structure J (it is induced from $B = \mathbb{C}P^l$) and we have that $\alpha(x, Jx) = 1$ for a unit vector x and $\alpha(x, y) = 0$ if y and x are perpendicular over $\mathbb{C}$.

If $n = 4k+1$ the $M_\rho = (M, g^\rho)$ do not admit equivariant Spin-structures because $\mathbb{C}P^{2k}$ is not spin. If $n = 4k+3$ there is one equivariant Spin-structure induced from a Spin-structure on $\mathbb{C}P^{2k+1}$. For $k = 0, 1, 2$, a lengthy but straightforward calculation gives:

$$\begin{aligned}\eta(D, S^3_\rho) &= -\frac{1}{6} + \frac{1}{12}\rho^2 - \frac{1}{6}\rho^4 , \\ \eta(D, S^7_\rho) &= -\frac{11}{360} + \frac{11}{90}\rho^2 - \frac{11}{60}\rho^4 + \frac{11}{90}\rho^6 - \frac{11}{360}\rho^8 , \\ \eta(D, S^{11}_\rho) &= -\frac{191}{30240} + \frac{191}{5040}\rho^2 - \frac{191}{2016}\rho^4 + \frac{191}{1512}\rho^6 - \frac{191}{2016}\rho^8 \\ &\quad + \frac{191}{5040}\rho^{10} - \frac{191}{30240}\rho^{12} .\end{aligned}$$

The first result was computed by Hitchin (see [11]) from the Dirac spectrum of the classical Berger spheres.

4 Adiabatic Limits

Consider the canonical variation of the metric on M as in the proof of the vanishing theorem. We want to compute the limit $\rho \to 0$ of $\eta(D_\zeta, g^\rho)$. The condition of theorem 2.2 holds for small ρ if the corresponding condition for the quotient manifold B is satisfied, because the scalar curvature of M converges to that of B as $\rho \to 0$. The following theorem gives the limit of integrals as in 2.3.

In general let K be the multiplicative sequence associated to a power series k (see [12] or [14]), f an arbitrary power series in one variable starting with 1, ∇ a connection on π^*E extending ∇^0 with first Chern form $c_1(\nabla) \in \Omega^2(DE;\mathbb{R})$ and $\beta \in \Omega^*(B;\mathbb{R})$ arbitrary.

Theorem 4.1

$$\lim_{\tau\to 0} \int_{DE} K(p(g^\tau_{DE}))\, f(c_1(\nabla))\, \pi^*\beta =$$
$$= \left\langle K(p(TB))\ \beta\ \frac{(k(y^2)\ f(y) - 1)}{y}\bigg|_{y=c_1(E)} \bigg| [B] \right\rangle .$$

For the signature operator $S(g^\tau_M)$ (see [1]) we get at once:

Corollary 4.2

$$\lim_{\tau\to 0} \eta(S(g^\tau_M))) = \left\langle L(p(TB)) \left(\frac{1}{\tanh(c_1(E))} - \frac{1}{c_1(E)} \right) \bigg| [B] \right\rangle - \mathrm{sign}\,(DE, M).$$

By Theorems 2.2 and 4.1 the Atiyah-Patodi-Singer index theorem applied to the manifold (DE, M) yields for $K = \hat{A}$, $f(x) = e^{x/2}$, $k(x) = x^{1/2}/(2\sinh(x^{1/2}/2))$ and $\beta = \mathrm{ch}\,(\zeta_B) e^{c_1(\xi(\alpha_B))/2}$:

Corollary 4.3

$$\lim_{\rho\to 0} \frac{\eta(D_\zeta, g^\rho) + \dim\ker\,(D_\zeta, g^\rho)}{2} =$$
$$= \left\langle \hat{A}(B)\, e^{c_1(\xi(\alpha_B))/2} \mathrm{ch}\,(\zeta_B) \left(\frac{e^{c_1(E)/2}}{2\sinh\,(c_1(E)/2)} - \frac{1}{c_1(E)} \right) \bigg| [B] \right\rangle \bmod \mathbb{Z}.$$

If in addition
$$s_B(b) > 4\, ||R^{\alpha_B} \otimes 1 + 1 \otimes R^{\zeta_B}||(b)$$
for all $b \in B$, then the identity holds in $\mathbb{R}$ i.e. without reducing modulo $\mathbb{Z}$ and $\lim_{\rho\to 0} \dim\ker\,(D_\zeta, g^\rho)$ is trivial.

This formula was also obtained by W. Zhang [18] relying on the work in [7]. It follows that these limits do not depend on the metrics and connections involved but can be computed from the bundles only.

Proof of Theorem 4.1: By the formulae (3.1) to (3.4) and (3.5) for the Riemannian curvature tensor on DE we have

$$\lim_{\rho\to 0} R^{DE}_{x,y} = \begin{pmatrix} \begin{matrix} 0 & z \\ -z & 0 \end{matrix} & ? \\ 0 & \pi^* R^B \end{pmatrix}$$

with

$$z := \frac{f''(\tau)}{f(\tau)}\ u \wedge v + 2f'(\tau)\ \pi^*\alpha\ .$$

The invariant polynomial P defining the Pontrjagin forms from the curvature tensor is $p(R) = \det\,(1 + \frac{R}{2\pi})$ and has the property that $p\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = p(A)p(C)$. Because of the multiplicativity of K we therefore have

$$\lim_{\rho\to 0} K(p(g^\rho_{DE})) = \pi^* K(p(g_B)) \wedge k(z^2/4\pi^2).$$

As $\rho \to 0$ the integral in the theorem converges to

$$\int_{DE} \pi^*(K(p(TB))\ \beta)\ k(z^2/4\pi^2) f(z/2\pi)$$
$$= \int_{DE} \pi^*(K(p(TB))\ \beta)(k(z^2/4\pi^2) f(z/2\pi) - 1)\ .$$

The last factor is divisible by $z/2\pi$. So we can perform integration along the fibre and get

$$\int_B K(p(TB))\beta \frac{k(c_1(E)^2) f(c_1(E)) - 1}{c_1(E)}\ ,$$

since

$$\pi_!(z^l) = l\pi_!\left(\frac{f''(\tau) f'(\tau)^{l-1}}{f(\tau)}\ u \wedge v \wedge (2\alpha)^{l-1}\right) = (2\alpha)^{l-1}$$

and α/π represents $c_1(E)$.

References

[1] M. F. Atiyah, V. K. Patodi and I. Singer, *Spectral asymmetry and Riemannian geometry I*, Math. Proc. Camb. Philos. Soc. 77, 1975, 43–69.

[2] M. F. Atiyah, V. K. Patodi and I. Singer, *Spectral asymmetry and Riemannian geometry II*, Math. Proc. Camb. Philos. Soc. 78, 1975, 405–432.

[3] M. F. Atiyah, V. K. Patodi and I. Singer, *Spectral asymmetry and Riemannian geometry III*, Math. Proc. Camb. Philos. Soc. 79, 1976, 71–99.

[4] M. F. Atiyah and I. Singer, *The index of elliptic operators III*, Ann. of Math. 87, 1968, 546–604.

[5] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Grundlehren der Mathematischen Wissenschaften, 298, Springer Verlag, Berlin and New York, 1991.

[6] A. L. Besse, *Einstein manifolds*, Springer Verlag, Berlin and New York, 1986.

[7] J.-M. Bismut and J. Cheeger, *η-invariants and their adiabatic limits*, J. Amer. Math. Soc. 2, 1989, 33–70.

[8] X. Dai, *Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence*, J. Amer. Math. Soc. 4 (1991), no. 2, 265–321.

[9] X. Dai and W.P. Zhang, *Circle bundles and the Kreck-Stolz invariant*, Trans. Amer. Math. Soc. 347 (1995), no. 9, 3587–3593.

[10] P. Gilkey, Invariance theory, *the heat equation, and the Atiyah-Singer index theorem*, Publish or Perish Pressm, Wilmington Delaware, 1984.

[11] N. Hitchin, *Harmonic spinors*, Adv. Math. 14, 1974, 1–55.

[12] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Grundlehren der mathematischen Wissenschaften Bd. 131, Springer, 1966.

[13] M. Kreck and S. Stolz, *Nonconnected moduli spaces of positive sectional curvature metrics*, J.-Amer.-Math.-Soc. 6 (1993), no. 4, 825–850.

[14] H. B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, 1989.

[15] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris, Sér. A-B 257, 1963, 7–9.

[16] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. 13, 459–469, 1966.

[17] J. Seade and B. Steer, *A note on the Eta function for quotients of $PSL_2(\mathbb{R})$ by co-compact Fuchsian groups*, Topology 26, 1987, 79–91.

[18] W. Zhang, *η-invariants and Rokhlin congruences*, C. R. Acad. Sci. Paris A 315, 1992, 305–308.