Infima of Universal Energy Functionals on Homotopy Classes

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Abstract

We consider the infima $\widetilde{E}(f)$ on homotopy classes of energy functionals E defined on smooth maps $f: M^n \to V^k$ between compact connected Riemannian manifolds. If M contains a submanifold L of codimension greater than the degree of E then $\widetilde{E}(f)$ is determined by the homotopy class of the restriction of f to $M \setminus L$. Conversely if the infimum on a homotopy class of a functional of at least conformal degree vanishes then the map is trivial in homology of high degrees.

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1 Introduction

The main objective of this paper is a correspondence between universal energy functionals on one side and factorization over subskeleta up to homotopy on the other. By an energy functional $E_\phi: C^\infty(M,V) \to \mathbb{R}^+$ we mean an integral $\int_M \phi(\nabla f, \ldots, \nabla^r f)$ of a constant coefficient differential operator ϕ over M, where M and V are smooth compact connected Riemannian manifolds, see section 2.

By a theorem of White, [6], the infimum

$$\widetilde{E}_\Phi(f) := \inf \{E_\Phi(g) | g: M \to V,\ g \simeq f\}$$

of a first order functional depends only on the homotopy class of the restriction of f to the d-skeleton M^d of a triangulation of M, where d is the degree of ϕ. Theorem 3.1 contains an analogous statement for functionals involving higher derivatives of f.

For first order functionals we consider a kind of converse, namely the implications of $\widetilde{E}_\phi(f) = 0$, restricting our considerations to functionals of (at least) conformal degree $d = n = \dim M$. For instance if $\phi(df) = \det(df^* df)^{1/2}$ and $\widetilde{E}_\phi(f) = 0$ then f is homotopic to a map $\tilde{f}: M \to V^{n-1}$ into the $n-1$-skeleton of a triangulation of V. For the n-norm $\phi(df) = \|df\|^n = \mathrm{Tr}\,((df^*df)^{n/2})$ it is proved in [6] that $\widetilde{E}_\phi(f) = 0$ implies that $f \simeq * = V^0$ is nullhomotopic.

Theorem 3.2 interpolates homologically between these facts. Since ϕ depends explicitely only on the differential df, the functional E_ϕ corresponds biuniquely to a symmetric function in n variables, see section 2. Fundamental are the Jacobians, i.e. the conformal powers $\phi(df) = \sigma_l^{n/2l}(df^*df)$ of the elementary symmetric polynomials σ_l in the eigenvalues of df^*df. Proposition 2.1 establishes a hierarchy $\sigma_1^{n/2} >> \sigma_2^{n/4} >> \cdots \sigma_l^{n/2l} >> \cdots >> \sigma_n^{1/2}$, where $\phi >> \psi$ indicates that we can estimate the energies $CE_\phi(f) \geq E_\psi(f)$ with a constant C independent of f. With respect to $>>$ the n-norm $\|df\|^n = \mathrm{Tr}\,((df^*df)^{n/2})$ is equivalent to $\sigma_1^{n/2}(df^*df) = (\mathrm{Tr}\,(df^*df))^{n/2}$. If $\phi >> \sigma_l^{n/2l}$ then $\widetilde{E}_\phi(f) = 0$ implies that f induces 0 in (co)homology of degree $\geq l$ with arbitrary coefficients, just like a map which factors over the $(l-1)$-skeleton $V^{l-1} \hookrightarrow V$ of some triangulation of $V^0 \subset V^1 \subset \cdots \subset V^{k-1} \subset V$.

If f is homotopic to a map $\tilde{f}\colon M \to V^{l-1} \subset V$ then clearly $\widetilde{E}_{\sigma_l^r}(f) = 0$ for all r. Actually we have $\widetilde{E}_\Phi(f) = 0$ for all polynomials Φ of length $\geq l$. I do not know in how far a converse holds. Theorem 3.2 only yields the vanishing of the primary obstructions in this factorization problem and this needs at least conformal degree $r \geq n/2l$. Proposition 3.3 shows that a general converse requires to assume even higher than conformal degree.

Rationally these obstructions can be expressed by integrals over M of certain forms constructed from minimal models, as in [2],[5] for instance. The energy functionals behave like an absolute value thereof, similiar to the case of mapping degree and volume for maps between manifolds of the same dimension $n = k$. Consider as an example the 2-Jacobian $\phi(A) = \sigma_2^{3/4}(A^*A)$ for maps $M^3 \to V^2$, i.e. $E_\phi(f) = \int_M |f^*\omega|^{3/2}$ where ω is the volume form of V. In [4] it is shown that the Hopf invariant of a map $M = S^3 \to S^2 = V$ can be estimated by $E_{\sigma_2^{3/4}}(f)$. Thus if $\widetilde{E}_{\sigma_2^{3/4}}(f) = 0$ then f is nullhomotopic. More generally the infinite higher homotopy groups of spheres are $\pi_{4k-1}(S^{2k}) = \mathbb{Z} \oplus G$, G finite, and the Hopf invariant $h_k : \pi_{4k-1}(S^{2k}) \to \mathbb{Z}$ detects the free part. As before one gets that $h_k(f) = 0$ if $\widetilde{E}_{\sigma_{2k}^{(4k-1)/2k}}(f) = 0$ but if $k > 1$ this does not imply that f is nullhomotopic. I am grateful to T. Rivière for this remark.

2 Energy functionals

For smooth maps $f\colon M^n \to V^k$ of compact Riemannian manifolds we consider functionals

$$E_\phi(f) := \int_M \phi(\nabla f, \nabla^2 f, \ldots, \nabla^r f) \ ,$$

of order r parametrized by functions

$$\phi : \bigoplus_{j=1}^{r} \left((\mathbb{R}^n)^{\otimes j} \otimes \mathbb{R}^k \right) \to \mathbb{R}_0^+ \ .$$

These functions are assumed invariant under the (diagonal) action of $O(n)\times O(k)$. They can therefore be evaluated on the derivatives $\nabla^j f \in \Gamma(TM^{*\otimes j} \otimes f^*TV)$ to yield well defined energy densities $\phi(\nabla f, \nabla^2 f, \ldots, \nabla^r f)$. We will say that ϕ (or E_ϕ) has degree $\leq d$ if it satisfies an estimate

$$|\phi(A_1, \ldots, A_r)| \leq C(1 + \|A_1\|^d + \|A_2\|^{d/2} + \cdots + \|A_r\|^{d/r}) \ , \ A_j \in (\mathbb{R}^n)^{\otimes j} \otimes \mathbb{R}^k \ .$$

Denoting by df^* the adjoint of df and using the Riemannian metrics to identify the dual tangent bundles we get a bundle endomorphism df^*df of TM. Examples of functionals as above are the classical (2-)energy, or more general p-energy $\int_M (\mathrm{Tr}\,(df^*df))^{p/2}$ of degree p, the exponential energy $\int_M e^{\mathrm{Tr}\,(df^*df)}$ of infinite degree, the Willmore energies $W(f) := \int_M \|\nabla df\|^2$ of degree 4, $W_0(f) := \int_M \|\nabla df(df^*df)^{-1}\|^2$. For an immersion W_0 is the L^2-norm of the mean curvature.

Functionals of first order, i.e. involving only first order derivatives, are conveniently described by nonnegative functions $\Phi\colon M(n \times n, \mathbb{R})^+ \to \mathbb{R}_0^+$ defined on nonnegative symmetric matrices and invariant under conjugation by $O(n)$. We have $\Phi(A^*A) = \phi(A)$ and

$$E_\phi(f) = E_\Phi(f) := \int_M \Phi(df^*df))d\mathrm{vol}_g \ .$$

We describe the estimates between polynomial energy functionals corresponding to the successive lifting problem for a map $f\colon M \to V$ over the skeleta of V.

An invariant function Φ on symmetric matrices is essentially the same as a symmetric function in the eigenvalues. For an arbitrary symmetric function Φ in n variables denote by

$$l(\Phi) := \min\{r \mid \Phi(x_1, \ldots, x_r, x_{r+1} = 0, \ldots, x_n = 0) \neq 0\} \ .$$

the length of Φ. If Φ is polynomial (eventually with real exponents) then $l(\Phi)$ is the length of the shortest monomial ocurring in Φ. By $((x_1^{i_1} \cdots x_s^{i_s}))$, for $i_1 \geq i_2 \geq \cdots \geq i_s$, $i_j \in \mathbb{R}_0^+$, we denote the symmetrization of $x_1^{i_1} \cdots x_s^{i_s}$, i.e. the function

$$\sum_{\pi \in S_n} x_{\pi 1}^{i_1} \cdots x_{\pi s}^{i_s} \ .$$

We have a simple hierarchy of the functionals $E_{((x_1^{i_1} \cdots x_s^{i_s}))}$:

Proposition 2.1 *1. If $1 \leq r < s \leq l \leq n$ and $i_s \geq \epsilon \geq 0$ then*

$$E_{((x_1^{i_1} \cdots x_l^{i_l}))}(f) \leq n! E_{((x_1^{i_1} \cdots x_r^{i_r+\epsilon} \cdots x_s^{i_s-\epsilon} \cdots x_l^{i_l}))}(f) \ .$$

2. For any $\mu > 0$, $r \geq 1$ we have

$$E_\Phi \leq \frac{1}{\mu^{r-1}} E_{\Phi^r} + \mu \,\mathrm{vol} M \ .$$

Proof: Let $a_1(x) \geq a_2(x) \geq \ldots \geq a_n(x)$ be the eigenvalues of $d_x f^* d_x f$, $x \in M$. Now

$$\begin{aligned} E_{((x_1^{i_1} \cdots x_l^{i_l}))}(f) &= \int_M \sum_{\pi \in S_n} a_{\pi 1}^{i_1} \cdot \cdots \cdot a_{\pi r}^{i_r} \cdots a_{\pi s}^{i_s} \cdots a_{\pi l}^{i_l} \\ &\leq n! \sum_{\pi \in S_n} \int_{a_{\pi 1} \geq \ldots \geq a_{\pi l}} a_{\pi 1}^{i_1} \cdot \cdots \cdot a_{\pi r}^{i_r} \cdots a_{\pi s}^{i_s} \cdots a_{\pi l}^{i_l} \\ &\leq n! \sum_{\pi \in S_n} \int_{a_{\pi 1} \geq \ldots \geq a_{\pi l}} a_{\pi 1}^{i_1} \cdot \cdots \cdot a_{\pi r}^{i_r + \epsilon} \cdots a_{\pi s}^{i_s - \epsilon} \cdots a_{\pi l}^{i_l} \\ &\leq n! E_{((x_1^{i_1} \cdots x_r^{i_r+\epsilon} \cdots x_s^{i_s - \epsilon} \cdots x_l^{i_l}))}(f) \end{aligned}$$

For the second claim we decompose the domain of integration to estimate

$$\begin{aligned} \int_M \Phi(df^* df) &\leq \int_{\{x | \Phi(d_x f^* d_x f) > \mu\}} \frac{\Phi(df^* df)^r}{\mu^{r-1}} + \int_{\{x | \Phi(d_x f^* d_x f) \leq \mu\}} \mu \\ &\leq \frac{1}{\mu^{r-1}} E_{\Phi^r} + \mu \mathrm{vol} M \ . \end{aligned}$$

•

We denote by $\sigma_l(A)$ the lth elementary symmetric polynomial in in the eigenvalues of a matrix A. It is determined by the identity $\det(1 + \lambda A) = \sum_l \sigma_l(A) \lambda^l$. If Φ and Ψ are homogeneous polynomials with $l(\Psi) \leq l(\Phi) = l$ then there are constants $\widetilde{C} = \widetilde{C}(\Psi, \Phi, n)$ and $C = C(\Phi, n, l)$ such that

$$\widetilde{C} C E_\Psi(f) \geq C E_\Phi(f) \geq E_{\sigma_l^{n/2l}}(f)$$

for all f.

3 Statement of Results

Theorem 3.1 *Let ϕ have degree d and let $L \subset M$ be a submanifold of codimension q with $q > d$. If $f, g \colon M \to V$ are homotopic on $M \setminus L$ then $\widetilde{E}_\phi(f) = \widetilde{E}_\phi(g)$.*

For instance in order to compute $\widetilde{W}(f)$ we may modify f in a tubular neighbourhood of a submanifold L of codimension at least 5. Even more specifically $\widetilde{W}(f) = 0$ for any map $S^n \to V$ if $n \geq 5$. It was proved by B. White in [6], see also Pluzhnikov [3], that for functionals of order 1 the same holds if f and g are homotopic on a $(q-1)$-skeleton of M with $q > d$, i.e. outside a skeleton of codimension $q > d$.

Theorem 3.2 *Assume that $\phi(A) \geq \sigma_l(A^*A)^{n/2l}$ and that $\widetilde{E}_\phi(f) = 0$. Then f induces 0 in (co)homology groups of degree $\geq l$.*

The following proposition shows that the assumptions of theorem 3.2 do not in general imply that f is homotopic to a map to $\tilde{f}: M \to V^{l-1} \subset V$.

Proposition 3.3 *Let $n = m + m'$, $k = l + l'$, $g: S^m \to S^l$, $g': S^{m'} \to S^{l'}$ and $f = g \times g': S^m \times S^{m'} \to S^l \times S^{l'}$. Assume that $r < m/2l$ or $r < m'/2l'$. Then $\widetilde{E}_\phi(g \times g') = 0 = \widetilde{E}_\phi(g \wedge g')$ for $\phi = \sigma_k^r$ where $g \wedge g': S^{m+m'} = S^m \wedge S^{m'} \to S^{l+l'}$ denotes the smash product of g and g'. In particular if $g: S^{n-1} \to S^{k-1}$ and $f = Sg = g \wedge id: S^m \to S^k$ is the suspension of g then $\widetilde{E}_\phi(f) = 0$ for $\phi = \sigma_k^r$ if $r < (n-1)/2(k-1)$.*

Proof: For the cartesian product $f = g \times g'$ we get

$$df^*df = \begin{pmatrix} dg^*dg & 0 \\ 0 & dg'^*dg' \end{pmatrix} ,$$

$$\sigma_k(df^*df) = \sigma_l(dg^*dg)\sigma_{l'}(dg'^*dg')$$

and

$$\widetilde{E}_\phi(f) = \widetilde{E}_{\sigma_l^r}(g)\widetilde{E}_{\sigma_{l'}^r}(g') \ .$$

If $r < m/2l$ or $r < m'/2l'$ then one of the factors vanishes by [6] or theorem 3.1, hence $\widetilde{E}_{\sigma_k^r}(f) = 0$.

The assertion about the smash product follows from that for the cartesian product. We deform g and g' to maps which are constant on some open sets $U \subset S^m$ and $U' \subset S^{m'}$ containing the base points. The projection $\pi: S^m \times S^{m'} \to S^m \wedge S^{m'}$ and its inverse have bounded differential over $S^m \setminus U \times S^{m'} \setminus U'$. Hence we can estimate

$$\begin{aligned} E_\phi(g \wedge g') &= \int\limits_{\pi(S^m \setminus U \times S^{m'} \setminus U')} \sigma_k^r(g \wedge g') \leq C \int\limits_{S^m \setminus U \times S^{m'} \setminus U'} \sigma_k^r(\pi' \circ (g \times g')) \\ &\leq C \int\limits_{S^m \times S^{m'}} \sigma_k^r(g \times g') = C E_{\sigma_{l'}^r}(g \times g') \end{aligned}$$

where $\pi': S^l \times S^{l'} \to S^l \wedge S^{l'}$. •

The generator of $\pi_4(S^3) \cong \mathbb{Z}/2$ is the suspension of the Hopf map $\eta: S^3 \to S^2$. Hence $\widetilde{E}_{\sigma_3^r}(s\eta) = 0$ for $r < 3/4$ but the conformal power is $r = 2/3$.

4 Proof of Theorem 3.1

Let R be small enough such that the exponential map restricted to the R-disc bundle of the normal bundle of L is a diffeomorphism, i.e. the map

$$\begin{array}{rcl} RD\nu(L) & \to & U_R(L) := \{x \in M \mid d(x,L) \le R\} \\ v \in \nu_p(L) & \mapsto & \exp_p(v) \ . \end{array}$$

We will write $\rho(x) = d(x,L)$ for the distance from L and $\mu(r,x) = \exp_p(rv)$ for $x = \exp_p(v) \in U_R(L)$ and $r \le R/\rho(x)$. Let $0 < \alpha < \beta = 2\alpha < \epsilon = 3\alpha < R$ and choose smooth functions $\chi\colon [0,R] \to [0,1]$ and $\psi\colon [0,R] \to [0,R/\alpha]$ such that $\chi(x) = 0$ for $x \in [0,\alpha]$, $\chi(x) = 1$ for $x \in [\beta,R]$, $\chi' \ge 0$ and $\psi(x) = R/\alpha$ for $x \in [0,\alpha]$, $\psi(x) = 1$ for $x \in [\epsilon,R]$, $\psi' \le 0$ and $R/x \ge \psi(x) \ge R/2x$ for $x \in [\alpha,\beta]$. If the functional in question has order r we also require that for the derivatives $\chi^{(j)}$ and $\psi^{(j)}$, $j = 0 \ldots r$, we have that $|\chi^{(j)}(x)| \le C/\alpha^j$ and $|\psi^{(j)}| \le C/\alpha^{j+1}$. Here and in the sequel C denotes a suitable constant independent of α. Denote by π the map $M \to M \times [0,1]$ given by

$$\pi(x) := (\mu(\psi(\rho(x)),x), \chi(\rho(x))) \ .$$

Note that the image of π is contained in the compact set

$$K := M \times \{1\} \cup M \setminus U_{R/2}(L) \times [0,1] \cup M \times \{0\} \ .$$

Let $H\colon K \to V$ be a smooth homotopy between $f = H|_{M\times\{0\}}$ and $g|_{M\times\{1\}}$ outside L. The map h defined by $h(x) := H(\pi(x))$ is homotopic to f and coincides with g on $M \setminus U_\epsilon(L)$. Thus

$$\widetilde{E}_\phi(f) - \widetilde{E}_\phi(g) = \widetilde{E}_\phi(h) - \widetilde{E}_\phi(g) = \int_{U_\epsilon} \phi(\nabla h, \ldots, \nabla^r h) - \int_{U_\epsilon} \phi(\nabla g, \ldots, \nabla^r g) \ .$$

For the proof of the theorem it suffices to show that the energy of h on $U_\epsilon(L)$ can be made arbitrarily small by choosing α appropriately. So we need to estimate the (covariant) derivatives of order $\le r$ of h. Up to some constants depending on the derivatives of H on the compact set K we have to estimate $\nabla^j\pi$, $j = 1 \ldots r$. At $p \in L$ we use a chart ξ of $U_R(L)$ mapping L to $\mathbb{R}^{n-q} \times \{0\}$, preserving ρ, i.e. $\xi(z) = (x,y)$ with $\rho(z) = |x|$, and such that $\xi(\mu(r,z)) = (rx,y)$. Up to constants depending on derivatives of ξ we have to estimate the derivatives of

$$(x,y) \mapsto (\psi(|x|)x, y, \chi(|x|)) \ .$$

A straightforward calculation shows that

$$\begin{array}{rcl} \nabla^j \chi(|x|) & = & \displaystyle\sum_{i=1}^{j} \chi^{(i)}(|x|)\mathcal{O}(1/|x|^{j-i}) \ , \\ \nabla^j \psi(|x|)x & = & \displaystyle\sum_{i=1}^{j} \psi^{(i)}(|x|)\mathcal{O}(1/|x|^{j-i-1}) \ . \end{array}$$

Hence

$$||\nabla^j \chi(|x|)|| \le C\alpha^{-j} \quad \text{and} \quad ||\nabla^j\left(\psi(|x|)x\right)|| \le C\alpha^{-j} \; .$$

Therefore we can estimate $\nabla^j \pi$ by α^{-j} and the energy integral by

$$\begin{aligned} E_\phi(h|_{U_\epsilon(L)}) &= \int_{U_\epsilon(L)} \phi(\nabla h, \ldots, \nabla^r h) \\ &\le \mathrm{vol} U_\epsilon(L) C(1 + |\nabla h|^d + \cdots + |\nabla^r h|^{d/r}) \\ &\le C\epsilon^q \alpha^{-d} = C3^q \alpha^{q-d} \end{aligned}$$

If $q > d$ then this tends to 0 for $\alpha \to 0$. •

5 Proof of Theorem 3.2

Let M^q, V^q denote the q-skeleta of smooth triangulations of M and V respectively. By the Deformation Theorem of geometric measure theory, [1], we can deform the restriction $f|_{M^q}$ of f to a Lipschitz map $\tilde{f}$ mapping M^q to V^q such that for the q-dimensional Hausdorff measures $\mathcal{H}$ we have

$$\mathcal{H}^q(\tilde{f}(M^q)) \le C_1 \mathcal{H}^q(f(M^q))$$

where C_1 does not depend on f.

It suffices to consider the case $\Phi = \sigma_l^{n/2l}$. We choose an embedding of the tangent bundle $TM \hookrightarrow M \times \mathbb{R}^s$ into a trivial bundle of rank s. Let π be the orthogonal projection $M \times \mathbb{R}^s \to TM$. Chose ϵ small enough so that $\Psi_v : m \mapsto \exp_m(\pi v)$ is a diffeomorphism of M for all $v \in D^s = \{v \in \mathbb{R}^s \mid |v| \le \epsilon\}$. Also let $U_\epsilon M^q$ denote the image of $M^q \times D^s$ under the map $\Psi : (m, v) \mapsto \exp_m(\pi v)$. Let $f_v(m) = f(\exp_m(\pi v))$. We denote by $e_\Phi(f) = \Phi(df^* df)$ the energy density and apply the transformation formula to the submersion $\Psi : M^q \times D^s \to U_\epsilon M^q$. Let $J_n \Psi$ be the n-Jacobian of this map,

$$C_2 = \frac{1}{\max_{z \in U_\epsilon M^q} \mathcal{H}^{s+q-n}(\Psi^{-1}(z))}$$

and

$$C_3 = \frac{\min_{M^q \times D^s} J_n \Psi}{\max_{z \in U_\epsilon M^q} \mathcal{H}^{s+q-n}(\Psi^{-1}(z)) \max_{m,v} |d_m \psi_v^* d_m \psi_v|^{n/2}} \; ,$$

where $|d_m \psi_v^* d_m \psi_v|$ denotes the operator norm of $d_m \psi_v^* d_m \psi_v$ for $m \in M^q$, $v \in D^s$, i.e. its largest eigenvalue. Then

$$\begin{aligned} E_\Phi(f) &\ge \int_{U_\epsilon M^q} e_\Phi(f) \\ &\ge C_2 \int_{U_\epsilon M^q} e_\Phi(f) \mathcal{H}^{s+q-n}(\Psi^{-1}(z)) dz \end{aligned}$$

$$
\begin{aligned}
&= C_2 \int_{M^q \times D^s} e_\Phi(f)(\Psi(y)) J_n \Psi(y) dy \\
&\geq C_2 \min_{M^q \times D^s} J_n \Psi \int_{M^q \times D^s} e_\Phi(f)(\Psi(y)) dy \\
&= C_2 \min_{M^q \times D^s} J_n \Psi \int_{D^s} \int_{M^q} e_\Phi(f)(\Psi_v(m)) dm dv \\
&\geq C_3 \int_{D^s} \int_{M^q} e_\Phi(f \circ \Psi_v) dm dv \ . \qquad (5.1)
\end{aligned}
$$

Assume that $E_\Phi(f) \leq \delta$ for some δ. By (5.1) we find $v \in D^s$ such that

$$\int_{M^q} e_\Phi(f_v|_{M^q})(m) dm \leq \delta / C_3 \ .$$

From proposition 2.1 there is a constant C_4 such that

$$
\begin{aligned}
\mathcal{H}^q(f_v(M^q)) &= E_{\sigma_q^{1/2}} \leq C_4 E_{\sigma_l^{q/2l}} \leq C_4 \left(\frac{1}{\mu^{n/q-1}} E_{\sigma_l^{n/2l}} + \mu \mathrm{vol} M \right) \\
&\leq C_4 \left(\frac{1}{\mu^{n/q-1}} \delta / C_3 + \mu \mathrm{vol} M \right)
\end{aligned}
$$

for any $\mu > 0$ provided $q \geq l$. Choosing δ and μ appropriately we can achieve that this is strictly less than the volume of the smallest q-cell of V^q. Thus we get a map $\tilde{f}_v$ homotopic to f which maps M^q to V^q and misses a point in every q-cell of V^q. Therefore it is homotopic to a map $M^q \to V^{q-1}$. •

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