HOMOTOPY CLASSES WITH SMALL JACOBIANS

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Abstract

If the infimum of the conformal k-Jacobian on the homotopy class of a map between compact Riemannian manifolds vanishes then the map factors rationally through the k-skeleton of the target manifold.

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1. Introduction

It follows from the Sobolev inequality that a map $f\colon M^m \to X$ between connected compact Riemannian manifolds M and X, $m = \dim(M)$, is nullhomotopic if its differential df has sufficiently small L^r-norm for some $r > m$. In fact, the diameter of the image of f is bounded by $\operatorname{diam}(\operatorname{im}(f)) \leq C\|df\|_r$ with some constant C depending only on M, X and $r > m$. In the conformal case $r = m$ this simple argument fails. A map f with arbitrary small $\|df\|_m$ can have arbitrarily large image. But by a theorem of White [10] there is a constant $\epsilon > 0$ depending on the geometries of M and X such that f is nullhomotopic if $\|df\|_m < \epsilon$.

We consider the analogous question for the Jacobian in place of the L^r-norm. For $k \in \mathbb{N}$ and $r \in \mathbb{R}^+$ these are the functionals

$$J_k^r\colon \mathcal{C}^\infty(M,X) \to \mathbb{R}_0^+ \ , \quad J_k^r(f) = \int_M \phi(df) \tag{1}$$

where

$$\phi(df) = |\underbrace{df \wedge \ldots \wedge df}_{k}|^{r/k} = \sigma(df^*df)^{r/2k}$$

and $\sigma_k(df^*df)$ denotes the kth elementary symmetric polynomial in the eigenvalues of df^*df. If $r = m = \dim(M)$ this functional is invariant under conformal changes of the metric on M.

In more general framework, for functionals $E\colon \mathcal{C}^\infty(M^m, X) \to \mathbb{R}_0^+$ we are interested in the information on the homotopy class of f detected by the infimum

$$\widetilde{E}\colon [M^m, X] \to \mathbb{R}_0^+ \ , \quad \widetilde{E}(f) := \inf\{E(g) | g\colon M^m \to X,\ g \simeq f\} \ ,$$

in particular in the consequences of $\widetilde{E}(f) = 0$.

We write $E_1 >> E_2$ if $\lim_\nu E_1(f_\nu) = 0$ implies $\lim_\nu E_2(f_\nu) = 0$ for any sequence $(f_\nu)_\nu$ in $\mathcal{C}^\infty(M^m, X)$. Among the Jacobians the Hölder inequality gives estimates

$$J_1^r >> J_2^r >> \cdots J_l^r >> \cdots >> J_m^r \tag{2}$$

and $J_l^{r_1} >> J_l^{r_2}$ if $r_1 \geq r_2$. With respect to $>>$, the Jacobian $J_1^r(f)$ is equivalent to the L^r-norm of the differential and $J_m^m >> \operatorname{vol}(\operatorname{im} f)$. If f is homotopic to a map $\tilde{f}\colon M \to X^{l-1} \subset X$ into the $(l-1)$-skeleton X^{l-1} of a triangulation of X we obviously have $\widetilde{J}_l^r(f) = 0$ for any r. The converse is known to hold in the extreme cases $l = 1$ and $l = m$ of (2) if $r \geq m$. It follows from a theorem of Pluzhnikov, [7], and White, [10], that $\widetilde{J}_l^r$ depends only on the restriction of f to the $[r]$-skeleton of a triangulation of M.

Let $f\colon M^m \to X$ be a map with $\widetilde{J}_l^m(f) = 0$. In [2] it is shown that f behaves homologically like a map into the $(l-1)$-skeleton X^{l-1} of a triangulation of X, i.e. induces 0 in homology of degree at least l. It is also shown there that f does not need to be homotopic to a map into X^{l-1}. Counterexamples produced in [2] arise from torsion elements in the higher homotopy groups of spheres. This suggests that f factors rationally. We prove:

Theorem 3. *Let X^{l-1} be the $(l-1)$-skeleton of a triangulation of X and assume that $\pi_1(X^{l-1}) = 0$. Let $f\colon M^m \to X$ be a map with $\widetilde{J}_l^m(f) = 0$. Then the rationalization $f_{\mathbb{Q}}\colon M \to X_{\mathbb{Q}}$ is homotopic to a map $f_{\mathbb{Q}}\colon M \to X_{\mathbb{Q}}^{l-1}$ into the rationalization of the $(l-1)$-skeleton of X.*

Remarks

(1) Theorem 3 extends a result of Rivière in [8] who showed that the Hopf invariant of a map $f\colon S^{4k-1} \to S^{2k}$ is estimated by $\widetilde{J}_{2k}^{4k-1}$. For maps between spheres the rational homotopy type is controlled by the Hopf invariant.
(2) For the Jacobians $J_l^r(f)$ with $l \geq 2$ and arbitrary large r one easily constructs surjective maps $M^m \to X^l$ with $\widetilde{J}_l^m(f) = 0$. Thus a simple argument based on a Sobolev-type inequality is not available in this case.

2. **Factorization in Rational Homotopy**

The proof of Theorem 3 is a computation in suitable relative Sullivan algebras, along the lines of [5], [6] where the number of homotopy classes of maps f was estimated by bounds on the dilatation. As before X^{l-1} denotes the $(l-1)$-skeleton of X. We denote by $X_{\mathbb{Q}}, X_{\mathbb{Q}}^{l-1}$ the rationalisations of X and X^{l-1} respectively. Thus we have maps $X \to X_{\mathbb{Q}}$ and $X^{l-1} \to X_{\mathbb{Q}}^{l-1}$ inducing isomorphisms $H^*(X, \mathbb{Q}) \cong H^*(X_{\mathbb{Q}}, \mathbb{Z})$ and $H^*(X^{l-1}, \mathbb{Q}) \cong H^*(X_{\mathbb{Q}}^{l-1}, \mathbb{Z})$. We assume that X^{l-1} is simply connected and $l \geq 2$. Then X is also simply connected and the above rationalisations are unique up to homotopy.

Let $\Omega(M)$, $\Omega(X)$ and $\Omega(X^k)$ denote the respective algebras of differential forms. By the functorial properties of Sullivan algebras the rationalization $f_{\mathbb{Q}}\colon M \to X_{\mathbb{Q}}$ is homotopic to a map $F\colon M \to X_{\mathbb{Q}}^{l-1} \subset X_{\mathbb{Q}}$ if there is a relative Sullivan algebra $\mathcal{S} := \Omega(X) \otimes_d \Lambda V \simeq \Omega(X^{l-1})$ and an extension $F^*\colon \mathcal{S} \to \Omega(M)$ of $f^*\colon \Omega(X) \to \Omega(M)$, see [3], [4]. We will first construct a suitable Sullivan algebra $\mathcal{S}$ and then use the estimate on the Jacobian of f to define F^*.

2.1. **Construction of $\mathcal{S}$**

We abbreviate $\mathcal{X} := \Omega(X)$, $\mathcal{Y} := \Omega(X^{l-1})$ and let $j\colon \mathcal{X} \to \mathcal{Y}$ be the morphism of commutative cochain algebras obtained by restriction. Up to homotopy we want to replace j by a morphism $\widetilde{j}$ into a relative Sullivan algebra $\mathcal{S}$ homotopy equivalent to $\mathcal{Y}$ such that the triangle

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\widetilde{j}"] \arrow[rr, bend right, "j"'] & \mathcal{S} = \mathcal{X} \otimes_d \Lambda V \arrow[r, "m", "\simeq"'] & \mathcal{Y} \end{tikzcd}$$

commutes up to homotopy. A relative Sullivan algebra ([4]) is a commutative cochain algebra $\mathcal{S} = \mathcal{X} \otimes_d \Lambda V$ such that there are graded vector spaces V_i, $i \in \mathbb{N}$,

$$V := \bigoplus_{i>0} V_i = \bigcup_{i>0} V(i)\ ,\ V(i) := V(i-1) \oplus V_i\ ,\ V(-1) := 0$$

and homomorphisms

$$d_i\colon V_i \to \mathcal{S}(i-1) := \mathcal{X} \otimes_d \Lambda V(i-1)$$

extending to a differential $d\colon \mathcal{S} \to \mathcal{S}$ which is nilpotent in the sense that

$$dV(i) \subset \mathcal{S}(i-1) := \mathcal{X} \otimes_d \Lambda V(i-1) \text{ where } S(-1) := \mathcal{X}\ .$$

The algebras $\mathcal{S}(q)$ are constructed together with morphisms $m_q\colon \mathcal{S}(q) \to \mathcal{Y}$ which are quasiisomorphisms in degrees increasing with q. To begin, let $V(0) \overset{\iota}{\hookrightarrow} \ker d \subset \mathcal{Y}$ be a vectorspace of cycles in $\mathcal{Y}$ such that $V(0) \oplus \operatorname{im} Hj$ generates $H(\mathcal{Y})$ as an algebra. Define

$$\mathcal{S}(0) := \mathcal{X} \otimes \Lambda V(0) \text{ with } d|_{V(0)} = 0$$

and

$$m_0\colon \mathcal{X} \otimes \Lambda V(0) \xrightarrow{j \otimes \Lambda\iota} \mathcal{Y}\ .$$

By construction m_0 induces an epimorphism $Hm_0\colon H(\mathcal{X}) \otimes \Lambda V(0) \to H(\mathcal{Y})$ in homology.

Proceeding by induction in q, assume $V(q)$ and

$$\mathcal{S}(q) := \mathcal{X} \otimes_d \Lambda V(q) \xrightarrow{m_q} \mathcal{Y}$$

have already been constructed such that Hm_q is surjective. Let $V_{q+1} \subset \mathcal{S}(q)$ with $V_{q+1} \cong \ker Hm_q \subset H(\mathcal{S}(q))$ be a vector space of representing cycles where the degree on V_{q+1} is set to be the degree inherited from of $\mathcal{S}(q)$ diminished by 1. The differential $d\colon V_{q+1} \to \mathcal{S}(q)$ is defined to be the inclusion $V_{q+1} \hookrightarrow \mathcal{S}(q)$. The map $m_{q+1}\colon V_{q+1} \to \mathcal{Y}$ is a lift of $m_q \circ d$ over $d^{\mathcal{Y}}$:

(1)
$$\begin{tikzcd} \ker Hm_q \arrow[r, hook] & H\mathcal{S}(q) \arrow[r, "Hm_q"] \arrow[d, hook] & H\mathcal{Y} \arrow[d, hook] \\ V_{q+1} \arrow[u, "\cong"] \arrow[r, hook, "d_{q+1}"] \arrow[drr, dashed, "m_{q+1}"'] & \mathcal{S}(q) \arrow[r, "m_q"] & \mathcal{Y} \\ & & \mathcal{Y} \arrow[u, "d^{\mathcal{Y}}"'] \end{tikzcd}$$

Extending m_{q+1} to $\Lambda V(q+1) = \Lambda(V_{q+1} \oplus V(q))$ we obtain $m_{q+1}\colon \mathcal{S}(q+1) := \mathcal{X} \otimes_d V(q+1) \to \mathcal{Y}$ which again induces an epimorphism in homology.

2.2. **Extension of f**

Let $f^*\colon \mathcal{X} = \Omega(X) \to \mathcal{M} = \Omega(M)$ and $j^*\colon \mathcal{X} = \Omega(X) \to \mathcal{Y} = \Omega(X^{l-1})$ be the morphisms of commutative cochain algebras induced by $f\colon M \to X$ and the inclusion $j\colon X^{l-1} \hookrightarrow X$ respectively. The homomorphism induced by j in cohomology is an isomorphism in degrees $< l-1$, injective in degree $l-1$ and 0 in degrees $> l-1$. Hence we may choose $V(0) \subset H^{l-1}(\mathcal{Y})$ such that

$$H(\mathcal{Y}) = V(0) \oplus \operatorname{im} Hj \ .$$

The morphism m_0 constructed as before then induces an isomorphism Hm_0 in degrees $\leq l-1$. Denote by $\mathfrak{i}(q)$ the ideal generated by $V(0)$ in $\mathcal{S}(q)$. In the diagram (1) we may split

$$V_{q+1} = V'_{q+1} \oplus d^{-1}_{q+1}\mathfrak{i}(q)$$

where V'_{q+1} lies in degrees $> l-1$.

We will inductively extend f^* to maps $f^*_q\colon \mathcal{S}(q) \to \mathcal{M}$ such that the f^*_q vanish on $\mathfrak{i}(q)$. To this end set

$$f^*_{-1} := f\colon \mathcal{S}(-1) = \mathcal{X} \to \mathcal{M}$$

and assume that we already have constructed the extension

$$f^*_q\colon \mathcal{S}(q) \to \mathcal{M}$$

satisfying estimates

$$\|f^*_q\omega\|_{n/r} \leq \epsilon\|\omega\|_{n/r} \tag{2}$$

for all $\omega \in \mathcal{S}(q)$ of degree r, $n \geq r > l-1$. Changing f by a homotopy we can have (2) with arbitrarily small value of ϵ.

For any r-cycle σ in M we find a homologous r-cycle σ' such that $\int_{\sigma'} |f^*_q\omega| < C_1 \int_M |f^*_q\omega|$ where C_1 does not depend on f^*_q (see [10], Proposition 3.1 for instance, or [2], proof of Theorem 3.2). If $d\omega = 0$ the Hölder inequality gives an estimate

$$\left|\int_\sigma f^*_q\omega\right| = \left|\int_{\sigma'} f^*_q\omega\right| \leq C_1\|f^*_q\omega\|_1 \leq C_1C_2\|f^*_q\omega\|_{n/r} \leq C_1C_2\epsilon\|\omega\|_{n/r}$$

with C_2 independent of f. In particular $f^*_q\omega$ is exact if $d\omega = 0$.

Thus $f^*_q(dV'_{q+1}) \subset d\mathcal{M}$. Let $\{\omega_j\}_j$ be a basis for V'_{q+1}. We define f^*_{q+1} on V_{q+1} by lifting f^*_q. More precisely, choose for each ω_j some $\alpha_j \in \mathcal{M}$ satisfying $d\alpha_j = f^*_q(d\omega_j)$ and the estimate (4) of the following Lemma 3. Define $f^*_{q+1}|_{\mathfrak{i}(q)} := 0$ and $f^*_{q+1}(\omega_j) := \alpha_j$.

Lemma 3. *Let M be a compact n-dimensional Riemannian manifold and denote by $\|\cdot\|_p$ the L^p-norm of differential forms given by the Riemannian metric. There is a constant $C \in \mathbb{R}$ depending on M (but not on β) such that for each exact r-form $\beta \in \Omega^r(M)$, $\beta \in d\Omega^{r-1}(M)$ there is $\alpha \in \Omega^{r-1}(M)$ with*

$$\beta = d\alpha \text{ and } \|\alpha\|_{n/(r-1)} \leq C\|\beta\|_{n/r} \ . \tag{4}$$

Proof: From the Hodge decomposition

$$\Omega(M) = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} d^*$$

the Laplacian Δ is invertible on $\operatorname{im} d \oplus \operatorname{im} d^*$. Let $\alpha := d^* \Delta^{-1} \beta$. Clearly $d\alpha = \beta$. Extending the operator Δ^{-1} on $\operatorname{im} d \oplus \operatorname{im} d^*$ by 0 to all of $\Omega^r(M)$ yields a bounded operator $L^p = W^{0,p} \to W^{2,p}$ into the Sobolev space $W^{2,p}$, [9]. Also $d^* \colon W^{2,p} \to W^{1,p}$ is bounded, [1]. From the Sobelev-embedding $\iota \colon W^{1,p} \subset L^{np/(n-p)}$ we infer that

$$\|\alpha\|_{np/(n-p)} \leq C \|\beta\|_p$$

where $C := \|\iota \; d^* \Delta^{-1}\|_{\mathrm{op}}$ is the operator norm. With $p = n/r$, $np/(n-p) = n/(r-1)$ the assertion follows. •

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