SOBOLEV SPACES: A SHORT COURSE

STEPHEN M. BUCKLEY
sbuckley@maths.may.ie

Department of Mathematics
National University of Ireland, Maynooth
Co. Kildare, Ireland.

0. Introduction

These notes are for the course in Sobolev spaces which I am giving to the Department of Mathematical Analysis during a visit to the University of La Laguna, Tenerife in May 2000. They will likely be neither a subset nor a superset of what I will actually present, but their Hausdorff distance from the presentation should be quite small!

Since this is a *very short* course, I have decided to slant the course somewhat towards one particular subarea, namely imbeddings of Sobolev spaces in spaces of measurable or continuous functions. Section 1 is motivational and not a core part of the course. Quite a few proofs are given for the basic results in Sections 2–3 but, where a proof is omitted, a standard reference is supplied. Sections 4 and 5 form a brief survey of two topics that particularly interest me. The references to the course, although numerous, are not meant to be comprehensive. Rather, they support the material presented in this course and give a flavour of some recent related research. I apologise in advance to the many people whose papers and books I have found enjoyable and enlightening and yet have not included in the reference list.

As I prepared these notes rather quickly, I found it convenient to use the excellent University of Jyväskylä course notes [H2] of Piotr Hajłasz as a basis for the development of the basic material in Sections 1–3. Also in view of the quick preparation, there are surely many little bugs throughout these notes for which I apologise in advance. If you have any comments or questions, please contact me at the above address or e-mail.

Let us also mention here some standing assumptions and some notation that will be used throughout the course. Ω is an open set in $\mathbb{R}^n$, $n > 1$, which may be subject to further explicit restrictions. If $E \subset \mathbb{R}^n$ and $t > 0$, $|E|$ and $\mathcal{H}^t(E)$ denote the Lebesgue and t-dimensional Hausdorff measure of E. Both $⨍_E u$ and u_E denote the average value of a function on a set E of positive measure (with respect to Lebesgue measure before Section 5, and with respect to μ in Section 5). D_i denotes the partial derivative in the ith direction and higher order partial derivatives are denoted as D^α, where α is a multi-index, i.e., an element of $\mathbb{Z}^n$ with non-negative coordinates. Given a point $x \in \mathbb{R}^n$, the symbols

Typeset by $\mathcal{AMS}$-TEX

$x_1, \dots, x_n$ are used to denote its Cartesian coordinates without further comment. $a \vee b$ is the larger of the two numbers a, b, and $a \wedge b$ the smaller.

1. Calculus of Variations and the Classical Dirichlet Problem

The classical Dirichlet problem for an open subset Ω of $\mathbb{R}^n$ is as follows: given $u \in C(\partial\Omega)$, find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \triangle u = 0 \text{ on } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

By the Calculus of Variations, solutions to $\triangle u = 0$ on Ω are essentially the same as minimizers of the Dirichlet integral $I(u) = \int_\Omega |\nabla u|^2$ (for instance this is true if $\partial\Omega$ is C^1, $u \in C^2(\overline{\Omega})$, and we minimize over $C^2(\overline{\Omega})$ functions).

The *Direct Method* in the Calculus of Variations depends on the following functional analytic result (and variations thereof); for more details, see for instance [E], [Gi], or [H2].

Theorem 1.1. *Suppose $I : X \to \mathbb{R}$, where X is a reflexive Banach space and I is convex, lower semicontinuous, and coercive. Then $I(x)$ attains its minimum value at some $u_0 \in X$. If I is strictly convex, then the minimum is unique.*

Definitions. In the above theorem, I is

- *convex* if $I(tu + (1-t)v) \le tI(u) + (1-t)I(v)$, $0 < t < 1$;
- *lower semicontinuous* if $I(u) \le \liminf_{n\to\infty} I(u_n)$, whenever (u_n) is norm-convergent to u;
- *coercive* if $I(u_n) \to \infty$ whenever $\|u_n\| \to \infty$.

To apply Theorem 1.1, we would like a reflexive Banach space that contains the smooth functions and on which the Dirichlet integral is lower semicontinuous. Now limits of smooth functions are not smooth, and it would be convenient to have I bounded on the space, so a first guess might be to use $\|u\| = \|\nabla u\|_{L^2(\Omega)}$ as a "norm". However at least when $|\Omega| < \infty$, constant functions indicate that this is not a norm. We therefore take as our Banach space essentially the closure of $C^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,2}$ given by the equation

$$\|u\|_{1,2}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2.$$

More precisely, we take the closure only of the set of $C^\infty(\Omega)$ functions whose $\|\cdot\|_{1,2}$-norm is finite. The resulting space $W^{1,2}(\Omega)$ is a reflexive Banach space.

As for the boundary values, we first define the functions in $W^{1,2}(\Omega)$ that have "zero boundary values", namely the subset $W_0^{1,2}(\Omega)$ consisting of the closure of the functions $C_0^\infty(\Omega)$. It is now convenient to replace the boundary function g with a function $w \in$

$W^{1,2}(\Omega)$; note that w is now defined on Ω rather than $\partial\Omega$ (but in some sense it has boundary values almost everywhere on Ω if Ω has some minimal smoothness).

Our substitute Dirichlet problem is now as follows: given $w \in W^{1,2}(\Omega)$, minimize $I(u) = \int_\Omega |\nabla(u+w)|^2$, over all $u \in W_0^{1,2}(\Omega)$. Now $W_0^{1,2}(\Omega)$ is also a reflexive Banach space since it is a closed subspace of a reflexive Banach space; see [R, Exercise 4.1]. It is clear that I is strictly convex (so any minimizer must be unique) and continuous on $W_0^{1,2}(\Omega)$, so once we prove coercivity, the existence of a (unique) minimizer follows. Coercivity follows immediately from the following inequality (sometimes called a *Poincaré inequality*); the space $W_0^{1,q}(\Omega)$ is defined in an analogous fashion to $W_0^{1,2}(\Omega)$.

Theorem 1.2. *Suppose $|\Omega| < \infty$ and $1 \le q < \infty$. Then for all $u \in W_0^{1,q}(\Omega)$,*

$$\int_\Omega |u|^q \, dx \le C|\Omega|^{q/n} \int_\Omega |\nabla u|^q \, dx, \tag{1.3}$$

where C depends only on n and q.

The Direct Method is well-known to PDE experts and can be found in many books and papers. There is much more to this story than is presented here: regularity of weak solutions, when are weak solutions classical solutions, other elliptic PDEs and PDE systems, etc. But this is a course on Sobolev spaces, not PDEs, so we will stop here with the advice that the reader should consult, for instance, [E] and [Gi] for more on this method. For more applications of Sobolev spaces to a large class of elliptic partial differential equations, see [HKM].

2. Sobolev spaces: Definitions and fundamental results

Suppose $u, v \in L^1_{\text{loc}}(\Omega)$. We say that $D^\alpha u = v$ *weakly*, where α is a multi-index, if for all $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega v\phi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi.$$

Weak derivatives are uniquely determined up to sets of measure zero. If $1 \le p \le \infty$, $k \in \mathbb{Z}$, $k \ge 0$, then the *Sobolev space* $W^{k,p}(\Omega)$ is the set of all $u \in L^p(\Omega)$ all of whose weak partial derivatives of order at most k exist and are in $L^p(\Omega)$. We also define

$$\|u\|_{k,p} \equiv \|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}.$$

If we are to be more honest, making $W^{k,p}(\Omega)$ a normed space requires the same trick as $L^p(\Omega)$, namely its elements must actually be taken to be equivalence classes of functions which agree almost everywhere. However we can usually gloss over this point.

If $u \in C^\infty(\Omega)$, then u has weak derivatives of all orders on Ω, namely its corresponding classical derivatives. It follows that $u \in C^\infty(\Omega)$ lies in $W^{k,p}(\Omega)$ if and only if all its classical derivatives of order at most k lie in $L^p(\Omega)$; in particular, this is the case if u is in addition compactly supported, i.e., it lies in $C_0^\infty(\Omega)$.

We define $W^{k,p}_{\rm loc}(\Omega)$ to be the class of functions in $W^{k,p}(\Omega)$ whose restrictions to compact subdomains Ω' of Ω lie in $W^{k,p}(\Omega')$. We also define $H^{k,p}(\Omega)$ to be the $\|\cdot\|_{k,p}$-closure of $C^\infty(\Omega)\cap W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$ to be the $\|\cdot\|_{k,p}$-closure of $C_0^\infty(\Omega)$. Our notation seems inconsistent with the last section since what we called $W^{1,2}(\Omega)$ there is called $H^{1,2}(\Omega)$ here. However we shall see that $W^{k,p}(\Omega)=H^{k,p}(\Omega)$ in general so this is not a problem.

Lemma 2.1. *$W^{k,p}(\Omega)$ is a Banach space (for all k,p,Ω).*

Proof. If (u_i) is a Cauchy sequence in $W^{k,p}(\Omega)$, then $(D^\alpha u_i)$ is also a Cauchy sequence in L^p whenever $|\alpha|\le k$. By completeness of L^p, there are functions u^α such that $D^\alpha u_i \xrightarrow{L^p} u^\alpha$ for each $|\alpha|\le m$. Now

$$\int_\Omega u\, D^\alpha\phi \longleftarrow \int_\Omega u_i\, D^\alpha\phi = (-1)^{|\alpha|}\int_\Omega D^\alpha u_i\,\phi \longrightarrow (-1)^{|\alpha|}\int_\Omega u_\alpha\phi.$$

Consequently $u_\alpha = D^\alpha u$ and $u_i \xrightarrow{W^{k,p}} u$. □

It follows that $W^{k,p}_0(\Omega)\subseteq H^{k,p}(\Omega)\subseteq W^{k,p}(\Omega)$, and that these are all Banach spaces.

Theorem 2.2. *$W^{k,p}_0(\mathbb{R}^n)=W^{k,p}(\mathbb{R}^n)$ for all $k\in\mathbb{N}$, $1\le p<\infty$.*

The case $p=\infty$ is very different: $W^{1,\infty}(\mathbb{R}^n)$ actually equals $\mathrm{Lip}(\mathbb{R}^n)$, and $x\mapsto|x|$ is a simple example of a function in $W^{1,\infty}(\mathbb{R})\setminus H^{1,\infty}(\mathbb{R})$.

Before proving Theorem 2.2, it is worthwhile recalling some facts about regularization; see [Z; Section 1.6] for proofs. Let $\phi\in C_0^\infty(\mathbb{R}^n)$ be non-negative, supported in the unit ball, and satisfy $\int_{\mathbb{R}^n}\phi=1$; for instance $\phi(x)=C\exp[-1/(1-|x|)^2]$ for $|x|\le 1$ gives one such function if C is appropriately chosen. Next we define $\phi_\epsilon(x)=\epsilon^{-n}\phi(\epsilon^{-1}x)$ and, given a function $u\in L^1_{\rm loc}(\mathbb{R}^n)$, we write $u_\epsilon=u*\phi_\epsilon$, where $*$ denotes convolution, and $\epsilon>0$. This collection of functions u_ϵ is often called a *regularization* of u; in particular we have $u_\epsilon\in C^\infty(\mathbb{R}^n)$. Notice that $u_\epsilon(x)$ is a weighted average of u over a ball of radius ϵ. In view of the Lebesgue differentiation theorem, it is thus not surprising that, as $\epsilon\to 0^+$, $u_\epsilon(x)\to u(x)$ almost everywhere and that, if in addition $u\in L^p(\Omega)$ and $1\le p<\infty$, then $u_\epsilon\xrightarrow{L^p}u$. Obviously, essentially the same results follow on a general open set Ω, the only twist being that u_ϵ is only defined on the set of points whose distance from $\partial\Omega$ is at least ϵ. Lastly, regularization interacts well with weak differentiation: $(D^\alpha u)_\epsilon = D^\alpha\, u_\epsilon$, as can be seen readily from the definition.

Another useful idea we need is the existence of a non-negative *bump function* $b\in C_0^\infty(\mathbb{R}^n)$ such that $b|_{B(0,1)}\equiv 1$ and $b|_{\mathbb{R}^n\setminus B(0,2)}\equiv 0$. Obviously it suffices to define such a function b_1 for $n=1$, as we can then define b in higher dimensions by the equation $b(x)=b_1(|x|)$. To define $b_1(t)$ for $t>0$, it suffices to integrate a one-dimensional regularization function ϕ as above between $-\infty$ and $3-2t$. Actually, it is not b itself that interests us, but rather the rescaled bump functions $b_R(x) = b(x/R)$. Note that b_R equals 1 on $B(0,R)$, zero outside $B(0,2R)$, and any kth order partial derivative of b_R has size no larger than C/R^k, where $C = C_{k,n}$ is independent of R.

Proof of Theorem 2.2. Suppose $u \in W^{k,p}(\Omega)$, and let (u_ϵ) be its regularization. Defining the scaled bump function b_R as above, we have that $b_R u_\epsilon \in C_0^\infty(\mathbb{R}^n)$ for all $\epsilon, R > 0$. By the properties of regularizations, it follows readily that we can pick a sequence (ϵ_n) such that $\|u - u_{\epsilon_n}\|_{W^{k,p}(\Omega)} < 1/n$. By the Dominated Convergence Theorem and the properties of bump functions, we can also pick a sequence (R_n) such that $\|u_{\epsilon_n} - b_{R_n} u_{\epsilon_n}\|_{W^{k,p}(\Omega)} < 1/n$. By the triangle inequality, it follows that $b_{R_n} u_{\epsilon_n} \overset{W^{k,p}}{\longrightarrow} u$, as required. $\square$

Theorem 2.3 (Meyers-Serrin). *If $u \in W^{k,p}_{\rm loc}(\Omega)$, $1 \le p < \infty$, then for every $\epsilon > 0$ there exists $v \in C^\infty(\Omega)$ such that $u - v \in W_0^{k,p}(\Omega)$ and $\|u - v\|_{k,p} < \epsilon$.*

Using Theorem 2.2, the proof is a routine partition of unity argument; see [H2; Theorem 11]. Note that this result implies that $H^{k,p}(\Omega) = W^{k,p}(\Omega)$.

It is not obvious from the definition when a function in $W^{k,p}(\Omega)$ lies in $W_0^{k,p}(\Omega)$. One useful criterion for bounded Ω is that if $u \in W^{1,p}(\Omega)$ is the restriction of a function $v \in C(\overline{\Omega})$, where $v|_{\partial\Omega} = 0$, then $u \in W_0^{k,p}(\Omega)$. The proof is not hard (as a hint, note that $v_\epsilon = (v - \epsilon) \vee 0$ is a compactly supported function for all $\epsilon > 0$, and has limit v as $\epsilon \to 0^+$).

Theorem 2.4. *If $1 < p < \infty$, $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are reflexive.*

Proof. Since a closed subspace of a reflexive space is reflexive, it suffices to note that $W^{1,p}(\Omega)$ (and hence $W_0^{1,p}(\Omega)$) is isomorphic as a Banach space with a closed subspace of $L^p(\Omega) \times \cdots \times L^p(\Omega)$ via the obvious map $u \mapsto (u, (D^\alpha u)_{|a| \le k})$, and that this product space is reflexive when $1 < p < \infty$. $\square$

2.5. ACL characterization of Sobolev spaces.

Recall that a continuous function $f : [a,b] \to \mathbb{R}$ is said to be *absolutely continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $I_1, \ldots I_k$ are pairwise disjoint intervals in $[a,b]$ with $\sum_{i=1}^k |I_k| < \delta$, we have $\sum_{i=1}^k |f(I_k)| < \epsilon$. If $G \subset \mathbb{R}$ is open, we say that u is absolutely continuous on G if it is absolutely continuous on all closed intervals in G. Absolutely continuous functions are differentiable almost everywhere.

The set of lines in a particular direction can be identified with the set of points in a hyperplane whose normal is in the same direction. Hence it makes sense to talk about properties that hold for *almost every line* parallel to a particular coordinate direction. We define $ACL(\Omega)$ to be the space of real-valued functions defined on Ω which are absolutely continuous on $L \cap \Omega$, for almost every line L parallel to any of the coordinate axes. Functions in this class were studied as far back as 1906 by Beppo Levi, and later by Tonelli. The interesting thing is that if we define $ACL^p(\Omega)$ to be the class of $ACL(\Omega) \cap L^p(\Omega)$ functions whose (classical) first order partial derivatives all lie in $L^p(\Omega)$ then:

Theorem 2.6 [Z; Theorem 2.1.4]**.** *$ACL^p(\Omega) = W^{1,p}(\Omega)$ (for all p, Ω).*

This statement is not quite honest since $ACL^p(\Omega)$ is a space of functions and $W^{1,p}(\Omega)$ a space of equivalence classes of functions which differ on sets of measure zero. More correctly, we mean that any function in $ACL^p(\Omega)$ lies in (an equivalence class in) $W^{1,p}(\Omega)$, while every element of $W^{1,p}(\Omega)$ has a representative in $ACL^p(\Omega)$. We shall ignore such pedantry in the rest of these notes.

Theorem 2.6 is quite useful. Let us consider a couple of applications, the first of which follows immediately from Theorem 2.6 together with the observation that $x \mapsto x \vee 0$ and $x \mapsto x \wedge 0$ are contraction mappings on $\mathbb{R}$.

Corollary 2.7. *If $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$, then $u_+ \equiv u \vee 0 \in W^{1,p}(\Omega)$. Furthermore*

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0, \\ 0, & u \le 0, \end{cases}$$

almost everywhere. A similar result applies to $u_- \equiv u \wedge 0$.

Before the second result, we first define a relatively closed set $E \subset \Omega$ to be *removable for* $W^{1,p}(\Omega)$ if $|E| = 0$ and $W^{1,p}(\Omega \setminus E) = W^{1,p}(\Omega)$ in the sense that every function in $W^{1,p}(\Omega \setminus E)$ can be approximated by the restrictions of functions in $C^\infty(\Omega)$. Theorem 2.6 now readily implies the following removability theorem for $W^{1,p}(\Omega)$ since if $\mathcal{H}^{n-1}(E) = 0$, then E is contained in a measure zero set of lines in a fixed direction (equivalently the projection of E onto a hyperplane also has $\mathcal{H}^{n-1}$-measure zero).

Theorem 2.8. *If $E \subset \Omega$ is relatively closed and $\mathcal{H}^{n-1}(E) = 0$ then E is removable for $W^{1,p}(\Omega)$.*

Theorem 2.8 is quite sharp. For instance if Ω is the unit ball $B(0,1)$ and E is the set $\{x \in B : x_1 = 0\}$ then $\mathcal{H}^{n-1}(E)$ is finite (and nonzero!) but E is clearly not removable since, using Theorem 2.6 yet again, it is easy to see that the function which is 1 on the upper half-plane and 0 on the lower half-plane is not in $W^{1,p}(\Omega)$. The fact that $B \setminus E$ is disconnected makes life easier, but is not crucial here: with only a little more effort we can show that $E' = E \cap B(0,1/2)$ is also not removable for $W^{1,p}(B(0,1))$.

3. The Representation Formula and the Sobolev Imbedding Theorem for $W_0^{1,p}(\Omega)$

The formula in the following basic lemma is usually called a *Representation Formula.* Generalizations of this type of inequality are used to help prove Sobolev-type imbeddings for much more general settings than the Euclidean-Lebesgue one we are examining e.g. the Heisenberg group and other Carnot groups, and more generally Carnot-Carathéodory spaces and other doubling metric measure spaces; see [BO; Section 6], [HaK; Sections 4,8], and [FLW]. In doubling metric measure spaces (metric d and measure μ) on which such an inequality holds, the integrand denominator $|x-z|^{n-1}$ below is replaced by something like $\mu(B(x,d(x,z)))/d(x,z)$, and the integration might be over a larger set than B.

Lemma 3.1. *If $B \subset \mathbb{R}^n$ is a ball and $u \in W^{1,1}(B)$, then for almost all $x \in B$ we have*

$$|u(x) - u_B| \le C_n \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy, \tag{3.2}$$

Proof. Since $W^{1,1}(B) = H^{1,1}(B)$, it suffices by a limiting argument to prove (3.2) for $u \in C^\infty(B)$. Fix $x \in B$ and let $y \in B$, $y \neq x$, $z = (y-x)/|y-x|$, and $\delta(z) = \max\{t > 0 : x + tz \in B\}$. Clearly

$$|u(x) - u(y)| \le \int_0^{\delta(z)} |\nabla u(x+sz)|\, ds.$$

Writing dz for surface measure on S^{n-1}, we now have

$$\begin{aligned}
|u(x) - u_B| &\le |B|^{-1} \int_B |u(x) - u(y)|\, dy \\
&= |B|^{-1} \int_{S^{n-1}} \int_0^{\delta(z)} t^{n-1} |u(x) - u(x+tz)|\, dt\, dz \\
&\le |B|^{-1} \int_{S^{n-1}} \int_0^{\delta(z)} t^{n-1} \int_0^{\delta(z)} |\nabla u(x+sz)|\, ds\, dt\, dz \\
&\le |B|^{-1} \int_{S^{n-1}} \int_0^{\delta(z)} \left(\int_0^{2r} t^{n-1}\, dt \right) |\nabla u(x+sz)|\, ds\, dz \\
&= C_n \int_{S^{n-1}} \int_0^{\delta(z)} |\nabla u(x+sz)| s^{1-n} \cdot s^{n-1}\, ds\, dz \\
&= C_n \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy.
\end{aligned}$$

Note that the first and last equalities are changes to and from polar coordinates. □

The Sobolev Imbedding Theorem is a name which is used to cover a variety of related imbeddings of Sobolev spaces into spaces of measurable or Hölder continuous functions. In this section, we use it to refer to the following theorem together with (3.9) and Theorem 3.10 which together cover the case $p \ge n/k$. When $p < n/k$, the number $p^* = np/(n-kp)$ which occurs in the next theorem is often called the *critical Sobolev exponent*, or simply the *Sobolev exponent*. The following theorem is essentially due to Sobolev [So1], [So2] in the case $p > 1$, and to Nirenberg [N] and Gagliardo [Ga] in the case $p = 1$; the proof given below is that of Nirenberg.

Theorem 3.3. *If $k \in \mathbb{N}$, $1 \le p < n/k$, and $p^* = np/(n-kp)$, then there exists a constant C, dependent only on k, n, and p, such that for all $u \in W_0^{k,p}(\Omega)$,*

$$\left(\int_\Omega |u(x)|^{p^*}\,dx\right)^{1/p^*} \le C\left(\int_\Omega |\nabla^k u(x)|^p\,dx\right)^{1/p} \tag{3.4}$$

Note 3.5. $|\nabla^k u|^2$ is the sum of squares of all kth order partial derivatives of u.

Note 3.6. By taking limits, it suffices to prove (3.4) for functions in $C_0^\infty(\Omega)$. Since such functions can be extended with zero values to be defined on all of $\mathbb{R}^n$, it suffices to prove this theorem in the case $\Omega = \mathbb{R}^n$ in which case $W_0^{1,p}(\Omega) = W^{1,p}(\Omega)$.[1] It is nevertheless convenient to state Theorem 3.3 for general Ω.

Proof of Theorem 3.3. In view of Note 3.6, it suffices to prove (3.4) for $u \in C_0^\infty(\mathbb{R}^n)$. We prove this first for $k = p = 1$; the general result will then follow without much additional effort.

By the Fundamental Theorem of Calculus, we see that for each i we have $|u(x)| \le \int_{-\infty}^\infty |D_i u|\,dt_i$, where the integrand is meant to be evaluated at the point whose ith coordinate is t_i, and whose jth coordinate, $j \ne i$, is x_j. Thus

$$|u(x)|^{n/(n-1)} \le \prod_{i=1}^n \left(\int_{-\infty}^\infty |D_i u|\,dt_i\right)^{1/(n-1)}.$$

We now integrate from $-\infty$ to ∞ with respect to each of the variables x_i in turn, each time applying Hölder's inequality to the $n-1$ factors on the right-hand side that do not depend on x_i. For instance after the first such step we get

$$\int_{-\infty}^\infty |u(x)|^{\frac{n}{n-1}}\,dx_1 \le \left(\int_{-\infty}^\infty |D_1 u|\,dt_1\right)^{\frac{1}{n-1}} \prod_{i=2}^n \left(\int_{-\infty}^\infty \int_{-\infty}^\infty |D_i u|\,dt_i\,dx_1\right)^{\frac{1}{n-1}}.$$

After all n steps, we obtain

$$\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)}\,dx \le \prod_{i=1}^n \left(\int_{\mathbb{R}^n} |D_i u|\,dx\right)^{1/(n-1)}.$$

Using the simple estimate $|D_i u| \le |\nabla u|$, we immediately deduce the case $p = 1$ of (3.4).

Suppose now that $1 < p < n$. Given $u \in C_0^\infty(\mathbb{R}^n)$, we define a non-negative function $v \in C_0^1(\mathbb{R}^n) \subset W_0^{1,p}(\Omega)$ by the equation $v^{1^*} = |u|^{p^*}$, i.e. $v = |u|^{p(n-1)/(n-p)}$. The case $p = 1$ of (3.4) now implies that

$$\int_{\mathbb{R}^n} |u|^{p^*} = \int_{\mathbb{R}^n} v^{1^*} \le C\left(\int_{\mathbb{R}^n} |\nabla v|\right)^{n/(n-1)}.$$

[1] By contrast, note that if $|\Omega| < \infty$, non-zero constant functions show that we do not get (3.4) for $u \in W^{1,p}(\Omega)$.

But $|\nabla v| = p(n-1)(n-p)^{-1}|\nabla u| \cdot |u|^{n(p-1)/(n-p)}$, so using Holder's inequality we see that

$$\int_{\mathbb{R}^n} |u|^{np/(n-p)} \le C' \left(\int_{\mathbb{R}^n} |\nabla u|^p \right)^{n/p(n-1)} \left(\int_{\mathbb{R}^n} |u|^{np/(n-p)} \right)^{n(p-1)/p(n-1)} .$$

Bringing the second factor over to the left-hand side, we deduce (3.4) for $k=1$, $p>1$.

Finally, we wish to prove the result for $k>1$, $p<n/k$. A little calculation shows that if we write $r = np/(n-(k-1)p)$, then $np/(n-kp) = nr/(n-r)$. Consequently, the case $k=m$ of (3.4) follows from a combination of the case $k=1$ for the function u and the case $k=m-1$ applied to the functions $D_i u$, $1 \le i \le n$; we leave the details to the reader. □

Note also that if $k=1$ and $p = nq/(n+q)$, then $p^* = q$. With this choice of parameters, (1.3) follows from Hölder's inequality applied to (3.4). More generally, assuming $|\Omega| < \infty$, if we rewrite (3.4) using averages, and then use Holder's inequality (possibly on both sides), we deduce the imbedding inequalities

$$\left(\fint_\Omega |u|^q \right)^{1/q} \le C_{k,n,p} |\Omega|^{k/n} \left(\fint_\Omega |\nabla^k u|^p \right)^{1/p}, \qquad u \in W_0^{k,p}(\Omega), \tag{3.7}$$

whenever $1 \le p, q < \infty$ and $(n-kp)q \le np$. This last inequality in turn can be restated in the form: $W_0^{k,p}(\Omega)$ continuously imbeds in $L^q(\Omega)$ whenever $|\Omega| < \infty$, $1 \le p, q < \infty$, and $(n-kp)q \le np$.

The point of the last paragraph was to show that (3.7) is true for $(n-kp)q < np$, and that it is a weaker imbedding than for the sharp exponent $q = np/(n-kp)$, $p < n/k$. However the Rellich-Kondrachov Theorem, which we now state, says that we get something in return for working with such non-sharp imbeddings.

Theorem 3.8 [Z; Theorem 2.5.1]**.** *If Ω is a bounded domain, $k \in \mathbb{N}$, $1 \le p < n/k$, and $0 < q < np/(n-kp)$, then $W_0^{k,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$.*

So far we have looked at Sobolev imbeddings of the form $W_0^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p < n/k$. For $kp = n$, it is not true that $W_0^{k,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ but, if $|\Omega| < \infty$, we at least get the weaker result $W_0^{k,p}(\Omega) \hookrightarrow \phi(L)(\Omega)$, where $\phi(x) = \exp(x^{p/(p-1)}) - 1$. The space $\phi(L)(\Omega)$, an Orlicz space which contains $L^p(\Omega)$ for every $p < \infty$, is defined as the space of functions for which the norm in (4.1) is finite. If you don't like Orlicz spaces, you can rewrite this imbedding as the following set of inequalities:

$$\left(\fint_\Omega |u(x)|^q \, dx \right)^{1/q} \le C q^{(p-1)/p} \left(\fint_\Omega |\nabla^k u(x)|^p \, dx \right)^{1/p}, \qquad u \in W_0^{k,p}(\Omega),\ kp = n,\ 0 < q < \infty, \tag{3.9}$$

where C depends on k, p, Ω, but crucially is independent of q. This limiting case of the Sobolev Imbedding Theorem was first proved by Trudinger [T] when $k = 1$. The more general result here follows from [Z; Theorem 2.9.1].

Finally in the case $kp > n$, functions in $W^{k,p}(\Omega)$ are continuous or, more precisely they have continuous representatives. For simplicity, we consider only the case $k = 1$, which dates back to Morrey [Mo]. The reader can deduce the behaviour in all except certain borderline cases when $k > 1$ by iterating this result (together with Theorem 3.3 if $p < n$); see also [Z; Chapter 2].

Theorem 3.10. *If $p > n$, and $u \in W_0^{1,p}(\Omega)$, then $u = \overline{u}$ almost everywhere for some Hölder continuous function $\overline{u}$ that satisfies*

$$|\overline{u}(x) - \overline{u}(y)| \le C_{p,n}|x-y|^{1-n/p}\|u\|_{1,p}, \qquad x, y \in \Omega. \tag{3.11}$$

Proof. As usual, it suffices to assume that $\Omega = \mathbb{R}^n$ and, by a limiting argument, it suffices to assume that $u \in C_0^\infty(\Omega)$, in which case $\overline{u} = u$ and (3.2) holds for all x in every ball. Let B be a ball containing x and y of radius $|x - y|$. Applying Hölder's inequality to (3.2), we see that

$$|u(x) - u_B| \le \left(\int_B |\nabla u|^p\right)^{1/p} \left(\int_B |x-z|^{(1-n)p/(p-1)}\right)^{(p-1)/p}$$

The second factor is easy to evaluate; after some algebra, it equals $C_{p,n}r^{1-n/p}/2$ for some constant $C_{p,n}$. Since we get a similar estimate for $|u(y) - u_B|$, the result follows. □

We next consider traces and extensions. Here we aim for simplicity rather than sharpness. First if $u \in W^{1,p}(\Omega)$, we define (for this paragraph) $\overline{u} : \Omega \to \mathbb{R}$ by the equation

$$\overline{u}(x) = \limsup_{r\to 0} \fint_{B(x,r)} u(z)\, dz$$

By the Lebesgue differentiation theorem, $\overline{u}$ is a representative of u in $W^{1,p}(\Omega)$. In fact it is a very nice representative, since as discussed in [H2, Theorem 44], there is a sequence of Hölder continuous functions u_k on Ω, and an increasing sequence of compact subsets X_k of Ω such that u_k and u agree on X_k, and the Hausdorff dimension of $\Omega \setminus \bigcup_k X_k$ is at most $n - p$. In that sense, we can talk about values of $W^{1,p}(\Omega)$ functions off an exceptional set of dimension $n - p$. More sharply, the exceptional set is of $W^{1,p}(\Omega)$-capacity zero; see [HKM, Chapter 4] for much more on this topic.

Since $\partial\Omega$ has dimension at least $n-1$ when Ω is bounded, it is thus reasonable to ask about the existence of a meaningful *trace operator* Tr for $W^{1,p}(\Omega)$, i.e. a linear operator from $W^{1,p}(\Omega)$ to some function space on $\partial\Omega$ containing $C^\infty(\overline{\Omega})$ with the property that $\operatorname{Tr} u = u|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$. In the opposite direction to trace operators are extension operators. We say that a linear operator $\mathrm{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ is an *extension operator* for $W^{1,p}(\Omega)$ if $Ef|_\Omega = f$ for all $f \in W^{1,p}(\Omega)$.

We now state a pair of simple trace and extension results which are essentially Theorems 4.3.1 and 4.4.1 of [EG], respectively.

Theorem 3.12. *Suppose Ω is a bounded Lipschitz domain and $1 \le p < \infty$. Let $\nu = \nu(z)$ denote the outward unit normal vector at any $z \in \partial\Omega$, and let us write $\langle \cdot, \cdot \rangle$ for the usual inner product on $\mathbb{R}^n$. Then there is a bounded trace operator* $\mathrm{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega, \mathcal{H}^{n-1})$ *which gives the following version of the Divergence Theorem:*

$$\int_\Omega f \ \operatorname{div} \Phi \, dx = -\int_\Omega \langle \nabla f, \Phi \rangle \, dx + \int_{\partial\Omega} \langle \Phi, \nu \rangle \operatorname{Tr} f \; d\mathcal{H}^{n-1},$$

for all C^1 functions $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, and all $f \in W^{1,p}(\Omega)$.

Theorem 3.13. *Suppose Ω is a bounded Lipschitz domain and $\Omega \subset\subset V$ for some open $V \subset \mathbb{R}^n$. Then for each $1 \le p < \infty$ there is a bounded extension operator* $\mathrm{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ *such that* $\mathrm{E} f$ *is supported on V for all $f \in W^{1,p}(\Omega)$.*

There are many excellent books that cover the basics of Sobolev spaces; see for example [A], [EG], [GT], [Ma], and [Z].

4. The Sobolev Imbedding Theorem for $W^{1,p}(\Omega)$

Inequalities (3.4) and (3.9) cannot hold for all functions in $W^{k,p}(\Omega)$, at least when $|\Omega| < \infty$, since non-zero constant functions provide easy counterexamples. Chapter 4 of Ziemer's book [Z] discusses in detail what are adequate substitutes for the zero boundary-value hypothesis. We shall examine only the case $k = 1$ in which case an adequate substitute is the assumption that $u_E = 0$ for some fixed $E \subset \Omega$ satisfying $0 < |E| < \infty$. In fact, we shall concentrate on the case of a bounded domain Ω, and functions satisfying $u_\Omega = 0$.

Just because we used the phrase "adequate substitute," it would be wrong to infer that given a bounded domain Ω there is some constant C such that (3.4) holds whenever $k = 1$, $1 \le p < n$, $u \in W^{1,p}(\Omega)$, $u_\Omega = 0$. Far from it!! We simply mean that with this assumption, there are no trivial counterexamples, so that if the inequality breaks down, it is instead because of the geometry of Ω. This makes Sobolev imbeddings for non-compactly supported functions much more fun, since unlike imbeddings for compactly supported functions, the validity of the imbedding is highly dependent on the geometry of the domain. An early and in-depth exploration of this complex analytic-geometric connection is to be found in [Ma]. We concentrate on more recent results that clarify the geometric nature of the answers.

As with $W_0^{1,p}(\Omega)$, the $W^{1,p}(\Omega)$ Sobolev imbeddings for a "nice" bounded domain $\Omega \subset \mathbb{R}^n$ depend on whether the exponent p is less than, equal to, or greater than n. In the case $1 \le p < n$, we say that Ω *supports a p-Sobolev-Poincaré inequality* if there is a constant C such that

$$\left(\int_\Omega |u - u_\Omega|^{np/(n-p)} \, dx \right)^{(n-p)/np} \le C \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}, \qquad u \in W^{1,p}(\Omega).$$

Sobolev [So1], [So2] proved that bounded domains $\Omega \subset \mathbb{R}^n$ satisfying a uniform interior cone condition support a p-Sobolev-Poincaré inequality whenever $1 < p < n$, while Gagliardo [Ga] and Nirenberg [N] independently proved the corresponding result for $p = 1$.

When $p = n$, we say that a bounded domain Ω *supports a Trudinger inequality* if the following Orlicz norm inequality holds for some C:

$$\|u - u_\Omega\|_{\phi(L)(\Omega)} \le C \left(\int_\Omega |\nabla u|^n \, dx \right)^{1/n}, \qquad u \in W^{1,n}(\Omega).$$

Here $\phi(x) = \exp(x^{n/(n-1)}) - 1$, and $\| \cdot \|_{\phi(L)(\Omega)}$ is the corresponding Orlicz norm on Ω defined by

$$\|f\|_{\phi(L)(\Omega)} = \inf \left\{ s > 0 \;\middle|\; \int_\Omega \phi\left(\frac{|f(x)|}{s} \right) dx \le 1 \right\}. \tag{4.1}$$

Trudinger [T] proved that bounded domains $\Omega \subset \mathbb{R}^n$ satisfying a uniform interior cone condition support a Trudinger inequality (although he did not use that term!).

Finally, suppose that $p > n$. Applying Theorem 3.10 and using bump functions to localize the support, it is easy to see that functions in $W^{1,p}(\Omega)$ have representatives that are continuous *everywhere*. Obviously the continuous representative is unique, so under the identification of elements of $W^{1,p}(\Omega)$ with their continuous representatives, we have $W^{1,p}(\Omega) \cap C(\Omega) = W^{1,p}(\Omega)$. We say that a bounded domain Ω *supports a p-Morrey inequality* if the following inequality holds for some C:

$$|u(x) - u(y)| \le C|x - y|^{1-n/p} \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}, \qquad u \in W^{1,p}(\Omega) \cap C(\Omega).$$

Given that Ω is bounded, this inequality is equivalent to the imbedding $W^{1,p}(\Omega) \hookrightarrow C^{0,1-n/p}(\overline{\Omega})$ (see [KR, Theorem 3.1]), where

$$||u||_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x \in \Omega} |u(x)| + \sup_{x,y \in \Omega, x \ne y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

This latter imbedding is more natural for unbounded domains (which we do not consider here). Morrey [Mo] showed that balls support a p-Morrey inequality for all $n < p < \infty$.

So much for the classical results. More recently, rather sharp geometric-analytic results have been found which first require us to define some geometric conditions.

We say that a domain $\Omega \subsetneq \mathbb{R}^n$ is a *uniform domain* if there exists $C > 0$ such that for every pair $x, y \in \Omega$, there is a path $\gamma : [0, L] \to \Omega$ parametrized by arclength with $\gamma(0) = x$, $\gamma(L) = x_0$ for which $L \le C|x - y|$, and $t \wedge (L - t) \le C\delta_\Omega(\gamma(t))$. Here and later, d_Ω denotes distance to $\partial\Omega$.

We say that a bounded domain Ω is a *John domain (with respect to* $x_0 \in \Omega$*)* if there exists $C > 0$ such that for every $x \in \Omega$, there is a path $\gamma : [0, L] \to \Omega$ parametrized by arclength such that $\gamma(0) = x$, $\gamma(L) = x_0$, and $d_\Omega(\gamma(t)) \ge t/C$, for all $t \in [0, L]$.

We say that a bounded domain is a *QHBC domain with respect to* $x_0 \in \Omega$ if there exists $C > 0$ such that for every $x \in \Omega$, there exists a path $\gamma = \gamma_x : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$, and

$$\int_\gamma \delta(z)^{-1}|dz| < C\left(1 + \log(1/\delta(x))\right)$$

Neither John nor QHBC depends strongly on x_0 in the sense that if they are true for one particular choice of x_0, they are true for all $x_0 \in \Omega$; however C will depend on x_0. It is not hard to show that all uniform domains are John domains and all John domains are QHBC domains, but that these implications cannot be reversed; a non-John QHBC domain is constructed in [GM]. For more on uniform and John domains, we refer the reader to [V2], [V3], and [NV]. The reader is referred to [SS1] and [K] for more on QHBC domains (also known in the literature as *Hölder domains*).

Suppose $0 < \alpha < 1$. A domain Ω is a *mean* α*-cigar domain* if there exists a constant C such that for every pair of points $x, y \in \Omega$, there exists a path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$, and

$$\int_\gamma \delta(z)^{\alpha-1}|dz| \le C|x-y|^\alpha.$$

This condition says that there is a path between x and y which satisfies the uniform (cigar) condition in some weak averaged sense. We refer the reader to [BK2], [GM], and [L] for more information about mean α-cigar domains; these domains are called "weak cigar domains" in [BK2] and "Lip_α extension domains" in [GM] and [L]. The last name derives from the fact that Ω is mean α-cigar if and only if all functions defined on Ω which are locally Lipschitz of order α are globally Lipschitz of order α; see [GM].

Our main Sobolev Imbedding result below also uses the notion of quasiconformal equivalence. We refer the reader to [BK2] for some basic facts about quasiconformal mappings, and to [V1] for a much more comprehensive introduction to their theory.

We now state our main result, which is a special case of Theorem 4.1 in [BK2].

Theorem 4.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain that is quasiconformally equivalent to a uniform domain (for instance any planar simply-connected domain).*

(i) *If* $1 \le p < n$*, then* Ω *supports a* p*-Sobolev-Poincaré inequality if and only if* Ω *is a John domain.*
(ii) Ω *supports the Trudinger inequality if and only if* Ω *is a QHBC domain.*
(iii) *If* $p > n$*,* Ω *supports a* p*-Morrey inequality if and only if* Ω *is a mean* α*-cigar domain for* $\alpha = (p-n)/(p-1)$*.*

The "if" part of (i) is due to Bojarski [B], while the "only if" part is due to Buckley and Koskela [BK1]. The "if" part of (ii) is due to Smith and Stegenga [SS2] while the "only if" part, together with all of (iii) are due to Buckley and Koskela [BK2].

The proofs of (ii), (iii) depend on one of a family of so-called slice conditions that are investigated further in [BO] and [BS1], where many other related imbeddings are proved. Slice conditions have also been used [BS2] to investigate what the quasiconformal image of a nice domain can look like. They are closely related to the concept of *Gromov hyperbolicity*, as explained in [BB]. Using a different approach (based on an inequality involving conformal capacity and quasihyperbolic distance), the case $p = n$ of Theorem 4.2 was improved in [Bu], where it is proved that statement (ii) above holds for all quasiconformal images of QHBC domains.

5. Sobolev spaces on metric measure spaces

A *metric measure space* (X, d, μ) is a triple such that (X, d) is a metric space and μ is a Borel measure on X; to avoid trivialities, we assume that balls have positive but finite measure. Of particular interest is the special case where μ is a *doubling measure*, i.e., a measure which satisfies the condition $\mu(B(x, 2r)) \le C\mu(B(x, r))$ for all metric balls $B(x, r)$, $x \in X$, $r > 0$; we then say that X is a *doubling space.*

There has been much investigation in recent years on Sobolev spaces $W^{1,p}(X)$. This is true in spite of the fact that until a few months ago, there was nothing in the literature about defining derivatives on any reasonable subclass of doubling spaces! This deficiency was rectified by Cheeger [C] but that is not our main concern here.

How do you define Sobolev spaces without using derivatives? There have been many definitions which are equivalent to each other in very many situations, but most of them have one central idea behind them: we do not need to define some version of ∇u but only a function g whose $L^p(X)$-norm in classical situations is comparable with that $|\nabla u|$. Below the notation $W^{1,p}(X)$ does not carry any specific meaning, but rather is a collective notation for all definitions of (first order) Sobolev spaces on a doubling space X. We use alternative notations such as $M^{1,p}(X)$ when referring to a specific Sobolev space definition.

The Banach space $M^{1,p}(X)$ is one of the early versions of $W^{1,p}(X)$. Due to Hajłasz [H1], $M^{1,p}(X)$ is the set of all functions $u \in L^p(X)$ such that

$$|u(x) - u(y)| \le d(x, y)(g(x) + g(y)), \qquad \text{a.e. } x, y \in X, \tag{5.1}$$

equipped with the norm

$$\|u\|_{M^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all g satisfying (5.1). It is quite useful in the study of extension operators. For instance, if $1 < p \le \infty$, there is a bounded extension operator

$$\mathrm{E} : W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

if and only if $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ [HM]. In particular by Theorem 3.13, $M^{1,p}(\Omega) = W^{1,p}(\Omega)$ whenever Ω is a bounded Lipschitz domain with Lebesgue measure attached, and $p > 1$.

The Banach space $N^{1,p}(X)$, another version of $W^{1,p}(X)$ defined by Shanmugalingam who termed it a *Newtonian space*, uses the concept of *upper gradients* in its definition. A Borel function $g : X \to [0, \infty]$ is an upper gradient of a measurable function $u : X \to \mathbb{R}$ if

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\, ds \tag{5.2}$$

for every rectifiable path $\gamma : [a, b] \to X$. Obviously we can take $g = |\nabla u|$ if $X = \mathbb{R}^n$ and u is smooth. Upper gradients were introduced by Heinonen and Koskela [HeK] and studied further in [KM].

It is tempting to define $N^{1,p}(X)$ as the collection of functions $f \in L^p(X)$ that have upper gradients $g \in L^p(X)$, equipped with the norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all upper gradients g of u. Unfortunately this definition is not conducive to good limiting processes, so instead we merely require $u \in N^{1,p}(X)$ to have a *p-weak upper gradient* in $L^p(X)$, and similarly take an infimum over $L^p(X)$ norms of p-weak upper gradients when defining $N^{1,p}(X)$. For the precise definition of a p-weak upper gradient, we refer the reader to [Sh2], but the idea is that we relax the upper gradient assumption, assuming that (5.2) is true only for *p-almost every rectifiable path*, which in turn involves the concept of the p-modulus of a path family (investigated in [V1] for example).

We say that X *supports a weak* $(1, p)$*-Poincaré inequality* if there exist constants $C > 0$ and $\tau \geq 1$ such that whenever $B = B(x, r)$ is a ball in X, and ρ is an upper gradient of a function u on $\tau B \equiv B(x, \tau, r)$, we have

$$⨍_B |u - u_B|\, d\mu \leq C \operatorname{diam}(B) \left(⨍_{\tau B} \rho^p \right)^{1/p}.$$

Notice that, by Hölder's inequality, if X supports a weak $(1, q)$-Poincaré inequality, it certainly supports a weak $(1, p)$-Poincaré inequality for all $p > q$.

Shanmugalingam [Sh1, Section 3.2] shows that $M^{1,p}(X) \hookrightarrow N^{1,p}(X)$, that $N^{1,p}(\Omega) = W^{1,p}(\Omega)$ when $X = \Omega$ is any Euclidean domain with Lebesgue measure attached, and that we essentially have $M^{1,p}(X) = N^{1,p}(X)$ if X is a doubling space that supports a weak $(1, q)$-Poincaré inequality for some $q < p$. Shanmugalingam also defines a space $N_0^{1,p}(E)$ for $E \subset X$ which is roughly the space of functions in $N^{1,p}(X)$ which vanish outside E and uses this to explore the concept of p-harmonic functions, defined as minimizers of certain energy integral. This investigation is continued in [KS], where for instance it is established that p-harmonic functions are Hölder continuous if the space is doubling and supports a weak $(1, q)$-Poincaré inequality for some $q < p$.

Using yet another definition of Sobolev spaces, $H^{1,p}(X)$, Cheeger's important new paper [C] shows that if X is a doubling space and supports a weak $(1, p)$-Poincaré inequality for some $p > 1$, then $H^{1,p}(X)$ is reflexive. Also under these assumptions, we can decompose X into a countable union of measurable sets U_α of positive measure, together with an exceptional null set N, such that each of the sets U_α can be "coordinatized" via a finite number of Lipschitz functions and such that, when restricted to any one set U_α, a Lipschitz function has "partial derivatives" almost everywhere in these coordinate directions, thus generalizing the well-known theorem of Rademacher on the differentiability almost everywhere of Lipschitz functions in a Euclidean context. Shanmugalingam [Sh1; Section 2.3] shows that in a general metric measure space, $H^{1,p}(X) = N^{1,p}(X)$ as long as $p > 1$.

There is far more to the theory of Sobolev spaces on a metric measure space, and the area is still developing quite fast. [HaK] is a good starting point for getting an overall picture of this field, as things stood in the middle of 1998. [KM] and [FHK] discuss the equivalence of different definitions of Sobolev spaces.

References

[A] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[BB] Z. Balogh and S.M. Buckley, *Geometric characterizations of Gromov hyperbolicity*, in preparation.

[B] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, Complex analysis, Joensuu 1987, Lecture Notes in Math. 1351, Springer, Berlin, 1989, pp. 52–68.

[Bu] S.M. Buckley, *Quasiconformal images of Hölder domains*, preprint.

[BK1] S.M. Buckley and P. Koskela, *Sobolev-Poincaré implies John*, Math. Research Letters **2** (1995), 577–593.

[BK2] S.M. Buckley and P. Koskela, *Criteria for Imbeddings of Sobolev-Poincaré type*, Internat. Math. Res. Notices (1996), 881–901.

[BO] S.M. Buckley and J. O'Shea, *Weighted Trudinger-type inequalities*, Indiana Univ. Math. J. **48** (1999), 85–114.

[BS1] S.M. Buckley and A. Stanoyevitch, *Weak slice conditions and Hölder imbeddings*, preprint.

[BS2] S.M. Buckley and A. Stanoyevitch, *Weak slice conditions, product domains, and quasiconformal mappings*, to appear in Rev. Mat. Iberoamericana.

[C] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Func. Anal. **9** (1999), 428–517.

[E] L.C. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS Regional Conference Series in Mathematics, vol. 74, Amer. Math. Soc., Providence, RI, 1990.

[EG] L.C. Evans and R.F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, New York, 1992.

[FLW] B. Franchi, G. Lu, and R.L. Wheeden, *A relationship between Poincaré-type inequalities and representation formulas in spaces of homogeneous type*, Internat. Math. Res. Notices (1996), 1–14.

[FHK] B. Franchi, P. Hajłasz and P. Koskela, *Definitions of Sobolev classes on metric spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1903–1924.

[Ga] E. Gagliardo, *Properieta di alcune classi di funzioni in piu variabli*, Ricerche Mat. **7** (1958), 102–137.

[GM] F.W. Gehring and O. Martio, *Lipschitz classes and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 203–219.

[Gi] M. Giaquinta, *Introduction to regularity theory for nonlinear elliptic systems*, Lectures in Mathematics ETH Zürich, Birkhauser, Basel, 1993.

[GT] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd edn., Springer, New York, 1983.

[H1] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), 403–415.

[H2] P. Hajłasz, *Sobolev spaces: Theory and applications*, Lecture notes, Summer school, Jyväskylä 1998.

[HaK] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000).

[HM] P. Hajłasz and O. Martio, *Traces of Sobolev functions on fractal type sets and characterization of extension domains*, J. Funct. Anal. **143** (1997), 221–246.

[HKM] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, Oxford, 1993.

[HeK] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–61.

[KS] J. Kinnunen and N. Shanmugalingam, *Quasi-minimizers on metric spaces*, preprint.

[K] P. Koskela, *Old and new on the quasihyperbolic metric,*, Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), Springer, New York, pp. 205–219.

[KM] P. Koskela and P. MacManus, *Quasiconformal mappings and Sobolev spaces*, Studia Math. **131** (1998), 1–17.

[KR] P. Koskela and F. Reitich, *Hölder continuity of Sobolev functions and quasiconformal mappings*, Math. Z. **213** (1993), 457–472.

[L] V. Lappalainen, *Lip_h-extension domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. Diss. **56** (1985), 1–52.

[Ma] V.L. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[Mo] C.B. Morrey, *Functions of several variables and absolute continuity, II*, Duke J. Math. **6** (1940), 187–215.

[NV] R. Näkki and J. Väisälä, *John disks*, Exposition. Math. **9** (1991), 3–43.

[N] L. Nirenberg, *On elliptic differential equations*, Ann. Scuola Norm. Pisa (III) **13** (1959), 1–48.

[R] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[Sh1] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Thesis, University of Michigan (2000).

[Sh2] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), 243–279.

[SS1] W. Smith and D.A. Stegenga, *Hölder domains and Poincaré domains*, Trans. Amer. Math. Soc. **319** (1991), 67–100.

[SS2] W. Smith and D.A. Stegenga, *Sobolev imbeddings and integrability of harmonic functions on Hölder domains*, Potential Theory (M. Kishi, ed.), de Gruyter, Berlin, 1991.

[So1] S.L. Sobolev, *On some estimates relating to families of functions having derivatives that are square integrable*, Dokl. Akad. Nauk SSSR **1** (1936), 267-270, (Russian).

[So2] S.L. Sobolev, *On a theorem of functional analysis*, Mat. Sbornik **46** (1938), 471–497, (Russian).

[T] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[V1] J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics 229, Springer, Berlin, 1970.

[V2] J. Väisälä, *Uniform domains*, Tôhoku Math. J. (2) **40** (1988), 101–118.

[V3] J. Väisälä, *Relatively and inner uniform domains*, Conf. Geom. Dyn. **2** (1998), 56–88.

[Z] W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer, New York, 1989.