SLICE CONDITIONS AND THEIR APPLICATIONS

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0. Introduction

Slice-type conditions and their relatives have been used in a number of papers of the author and various collaborators to investigate a variety of problems in analysis including Sobolev-type imbedding theorems ([BuKo], [BuOS], [BuSt1], [Bu]), quasiconformal equivalence of domains ([BuSt2], [Bu]), and Gromov hyperbolicity ([BaBu]). Despite their usefulness, these conditions are still somewhat mysterious, but some recent research has helped to shed more light on them ([BaBu], [BuSt3], [BuDiSt]).

There are quite a few varieties of slice conditions, partially reflecting the fact that different applications require different conditions. In this paper, we define some of these variants, discuss how they differ from each other, and summarize their applications. Along the way, we encounter quite a few examples, counterexamples, and open questions. We discuss (strong) slice conditions and Gromov hyperbolicity in Section 2, weak slice conditions and their applications in Section 3, and some examples and useful lemmas in Section 4.

1. Preliminaries

The spherical metric σ on $\mathbb{R}^n$ is the Riemannian metric with density $\rho(z) = 2/(1+|z|^2)$. The *Riemann sphere* $\overline{\mathbb{R}^n}$ is the metric completion of $(\mathbb{R}^n, \sigma)$ obtained by adding the single point ∞. We also denote the extension of σ to $\overline{\mathbb{R}^n}$ by σ; note that $\overline{\mathbb{R}^n}$ is a compact metric space of diameter π.

Throughout this paper, Ω is either a proper subdomain of $\mathbb{R}^n$ or of $\overline{\mathbb{R}^n}$, $n > 1$; in most cases it is the former, so that is the default. Also, $\delta(x)$ is the Euclidean distance from x to $\partial\Omega$, and $\Gamma(x, y)$ is the class of rectifiable paths $\lambda : [0, t] \to \Omega$ for which $\lambda(0) = x$ and $\lambda(t) = y$. As well as the Euclidean metric, we use two other metrics on Ω: the *inner Euclidean distance* and *quasihyperbolic distance*, defined respectively by

$$l(x,y) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma ds, \qquad k(x,y) = \inf_{\gamma \in \Gamma(x,y)} \int_\gamma \frac{ds(z)}{\delta(z)}, \qquad x, y \in \Omega.$$

where ds is length (one-dimensional Hausdorff) measure with respect to the Euclidean metric. We write $l_\Omega(x, y)$ and $k_\Omega(x, y)$ if the domain Ω needs to be specified.

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We can analogously define spherical counterparts $l^\sigma(x,y)$ and $k^\sigma(x,y)$. The Euclidean length and diameter of $S \subset \Omega$ are denoted $\text{len}(S)$ and $\text{dia}(S)$, respectively. The corresponding quantities for the quasihyperbolic metric are denoted $\text{len}_{k;\Omega}(S)$ and $\text{dia}_{k;\Omega}(S)$, or simply $\text{len}_k(S)$ and $\text{dia}_k(S)$. We write $B(x,r)$ and $B_l(x,r)$ for, respectively, the open Euclidean ball and open inner Euclidean ball, of radius r about x. For any of these quantities, a σ superscript indicates that we are using the spherical version. We do not distinguish notationally between paths and their images.

We denote by $[x,y]$ any k^σ geodesic between $x,y \in \Omega$; the parametrization is unimportant. If γ is a path that passes through points x and y, then $\gamma[x,y]$ denotes any segment of γ that has x,y as endpoints (there is an element of choice involved if γ passes through x or y more than once).

For any two numbers a, b, $a \vee b$ and $a \wedge b$ denote their maximum and minimum, respectively. Let $C \geq 1$ and let d be the Euclidean metric. A domain Ω is a *C-uniform domain* if, for every $x,y \in \Omega$, there is a path $\gamma \in \Gamma(x,y)$ of length L and parametrized by arclength for which $L \leq Cd(x,y)$, and $t \wedge (l-t) \leq C\delta(\gamma(t))$, $0 \leq t \leq L$. If we instead take d to be the inner Euclidean metric, we get an *inner C-uniform domain*. Uniform domains include all bounded Lipschitz domains, as well as some domains with fractal boundary, such as the interior of a von Koch snowflake. All uniform domains are inner uniform, and a slit disk is a standard example of an inner uniform domain that is not uniform. For more on (inner) uniform domains, see for instance [GeOs], [Ge], [Va1], and [Va2].

Let $C \geq 1$ and let $x_0 \in \Omega$ be fixed. We say that Ω is a *C-Hölder domain (with respect to x_0)* if, for all $x \in \Omega$, $k(x,x_0) \leq C \log(C/\delta(x))$. All bounded inner uniform domains are Hölder, but the converse is false. Gehring and Martio introduced Hölder domains and showed that they are bounded [GeMa, 3.9]; for more on Hölder domains, see [SmSt1] and [Ko].

2. Slice conditions and Gromov hyperbolicity

We begin by recalling the slice condition in [BaBu]. Given $x,y \in \Omega$ and a path $\gamma \in \Gamma(x,y)$, we write $(x,y;\gamma) \in \mathit{slice}(C)$ if there exist pairwise disjoint open subsets $\{S_i\}_{i=0}^m$ of Ω, $m \geq 0$, with $d_i \equiv \text{dia}(S_i) < \infty$ such that

$$\forall\, 0 < i < m,\ \ \forall\, \lambda \in \Gamma(x,y): \quad \text{len}(\lambda \cap S_i) \geq d_i/C; \tag{Sli$_1$}$$

$$\forall\, 0 \leq i \leq m: \quad \text{dia}_k(\gamma \cap S_i) \leq C; \tag{Sli$_2$}$$

$$\text{len}\left(\gamma \setminus \bigcup_{i=0}^m S_i\right) = 0; \tag{Sli$_3$}$$

$$B(x,\delta(x)/C) \subset S_0, \quad B(y,\delta(y)/C) \subset S_m. \tag{Sli$_4$}$$

We say that Ω is a *(two-sided) slice domain* if there exists a constant C such that $(x,y;\gamma) \in \mathit{slice}(C)$ for all $x,y \in \Omega$ satisfying $k(x,y) \geq \log 2$, and all quasihyperbolic geodesics $\gamma \in \Gamma(x,y)$. We say that Ω is a *(two-sided) inner slice domain* if it satisfies the variant of the above condition with d_i being the inner diameter of S_i.

Mainly in this paper, we stick to the Euclidean case, but we sometimes need to switch to the spherical versions of our metrics. We then talk about (inner) spherical slice domains. Note that inner slice implies slice; this is true whether we are using the Euclidean or spherical versions. The converse is false, as we discuss in Secction 4. It follows from the results in [BaBu, Section 4] that an inner spherical slice domain satisfies all the other spherical slice-type conditions in the literature.

The adjective *two-sided* is used in the above definitions for emphasis whenever we want to contrast these with one-sided conditions. We say that a condition defined in terms of properties of pairs of points x, y is *two-sided* if it holds uniformly over all $x, y \in \Omega$, or *one-sided* if it only holds uniformly over all $x \in \Omega$ but with $y = x_0$ fixed. It is not hard to construct one-sided (inner) slice domains that are not two-sided (inner) slice domains.

To state one of the main results in [BaBu], we need to define three other conditions linking the inner spherical and quasihyperbolic metrics on a spherical domain $\Omega \subsetneq \overline{\mathbb{R}^n}$. First, we say that Ω is *Gromov hyperbolic with respect to the quasihyperbolic metric* (or, more briefly, *kG-hyperbolic*) if there exists a constant C such that

$$\forall\, x,y,z \in \Omega \;\; \forall\, [x,y],\; [x,z],\; [y,z] \;\; \forall\, w \in [x,y]: \quad k^\sigma(w,[x,z]\cup[z,y]) \le C. \tag{Hyp}$$

The notion of Gromov hyperbolicity, which makes sense in general metric spaces, was conceived in the setting of geometric group theory [Gr1], [Gr2], [GhHa]. For the quasihyperbolic metric this property was extensively studied in [BoHeKo].

Next, $\Omega \subsetneq \overline{\mathbb{R}^n}$ is a *Gehring-Hayman domain* if there exists a constant C such that

$$\forall\, x,y \in \Omega \;\; \forall\, [x,y]: \qquad \operatorname{len}^\sigma([x,y]) \le C l^\sigma(x,y), \tag{GH}$$

Finally, $\Omega \subsetneq \overline{\mathbb{R}^n}$ is a *ball-separation domain* if there exists a constant C such that

$$\forall\, x,y \in \Omega \;\; \forall\, [x,y] \;\; \forall\, w \in [x,y] \;\; \forall\, \lambda \in \Gamma(x,y): \quad \lambda \cap B^\sigma(w, C\delta(w)) \ne \emptyset. \tag{BS}$$

The following theorem is a special case of one of the main results of [BaBu]; see Theorem 0.1 and the subsequent discussion in that paper.

Theorem 2.1. *The following conditions are quantitatively equivalent for* $\Omega \subsetneq \overline{\mathbb{R}^n}$*:*

(1) Ω *is kG-hyperbolic;*
(2) Ω *is an inner spherical slice domain;*
(3) Ω *is both a Gehring-Hayman and a ball separation domain.*

Theorem 2.1 makes it easier to check if the kG-hyperbolic and inner slice conditions hold, since (3) is more easily verified than the other two conditions. Of course if Ω is a bounded domain, we can use the Euclidean metric in place of the spherical metric.

Theorem 2.1 also has implications for quasiconformal invariance since Gromov hyperbolicity is well-known to be quasiconformally invariant. Its equivalence to the other two conditions provides the only known proof of the quasiconformal invariance of those other conditions. By contrast, it is not hard to construct examples to show that neither the Gehring-Hayman condition by itself nor the one-sided (inner) slice condition are quasiconformally invariant.

3. Weak slice conditions and their applications

3.1. Weak slice conditions.

We begin by defining the weak slice condition introduced by the author and O'Shea [BuOS]. This condition is an endpoint of a one-parameter family of weak slice conditions later considered in [BuSt1] and [BuSt2], but here we discuss only the original condition.

Suppose $C \geq 1$. A finite collection $\mathcal{F}$ of pairwise disjoint open subsets of Ω is a *set of (inner) C-wslices* for $x, y \in \Omega$ if

$$\forall\, S \in \mathcal{F} \;\; \forall\, \lambda \in \Gamma(x,y): \quad \mathrm{len}(\lambda \cap S) \geq d_S/C, \tag{WS-1}$$

$$\forall\, S \in \mathcal{F}: \; S \cap B(x, \delta(x)/C) = S \cap B(y, \delta(y)/C) = \emptyset \tag{WS-2}$$

where $d_S < \infty$ is the (inner) Euclidean diameter of S. Next, we define $\mathrm{WS}(x, y; C)$ by

$$\mathrm{WS}(x, y; C) = 1 + \sup\{\mathrm{card}(\mathcal{F}) \,:\, \mathcal{F} \text{ is a set of } C\text{-wslices for } x, y\}$$

and $\mathrm{WS}_{\mathrm{in}}(x, y; C)$ is defined analogously, using sets of inner C-wslices. Note that $1 \leq \mathrm{WS}_{\mathrm{in}}(x, y; C) \leq \mathrm{WS}(x, y; C)$, since the empty set is trivially a set of (inner) C-wslices. We use subscript notation such as $\mathcal{F} = \{S_i\}_{i=0}^m$ only in cases where we know that $\mathcal{F}$ is nonempty.

According to [BuSt3, Corollary 2.9], $\mathrm{WS}(x, y; C) \leq C(1 + k(x, y))$. To define a wslice condition, we reverse this last inequality, i.e. the pair x, y *satisfies the C-wslice condition* if

$$k(x, y) \;\; \leq C\, \mathrm{WS}(x, y; C), \tag{WS-3}$$

We refer to $(x, y, \mathcal{F})$ as a C-wslice dataset if $\mathcal{F}$ is a set of C-wslices for x, y, and $k(x, y) \leq C(1 + \mathrm{card}(\mathcal{F}))$. We say that Ω is a *(two-sided) C-wslice domain* if all pairs of points in Ω satisfy a C-wslice condition. Inner, one-sided, and spherical variants are defined in the obvious manner. Clearly all of these weak slice-type conditions are implied by their strong slice-type counterparts.

Although not explicitly assumed, the collection of weak slices given by a C-wslice condition for a pair of points x, y, $k(x, y) > 2C$, cover at least some fixed fraction of the quasihyperbolic length of any quasihyperbolic geodesic $[x, y]$; this follows from Lemma 4.4. This relaxation of (Sli_3) is the most important of the differences between any strong-type slice condition and the corresponding weak-type slice condition.

Consider the following condition on sets of (inner) wslices:

$$\forall\, S \in \mathcal{F} \;\; \exists\, z_S \in S: \quad B(z_S, d_S/C) \subset S. \tag{WS-5}$$

Although this condition may fail for a given set of (inner) wslices, it can nevertheless be assumed without loss of generality according to [BuSt3, Theorem 2.14], as long as we are willing to tolerate changing the set of wslices and making a controlled change in the value of the slice parameter C. Condition (WS-5) is sometimes useful for getting upper bounds on the number of slices in certain parts of Ω based on the number of balls of a given size that can be packed there.

The following theorem provides us with a large collection of slice domains that includes all simply-connected planar domains.

Theorem 3.2 [BuSt2, Theorem 3.1]**.** *A quasiconformal image of an inner uniform domain is an inner slice domain, quantitatively.*

3.3. Applications to Sobolev imbeddings.

A *Trudinger domain* is a domain $\Omega \subset \mathbb{R}^n$ of finite volume for which

$$\|u - u_\Omega\|_{\phi(L)(\Omega)} \le C \left(\int_\Omega |\nabla u|^n \, dx \right)^{1/n}, \qquad \text{for all } u \in C^1(\Omega),$$

where $\phi(t) = \exp(t^{n/(n-1)}) - 1$, and $\| \cdot \|_{\phi(L)(\Omega)}$ is the corresponding Orlicz norm on Ω defined by

$$\|f\|_{\phi(L)(\Omega)} = \inf \left\{ s > 0 \;\middle|\; \int_\Omega \phi(|f(x)|/s) \, dx \le 1 \right\}.$$

Smith and Stegenga [SmSt2] proved that all Hölder domains are Trudinger domains. This is fairly sharp since Koskela and the author [BuKo] subsequently proved that any Trudinger domain satisfying a variant of our slice condition is a Hölder domain. Thus Theorem 3.2 tells us that Hölder is equivalent to Trudinger for simply-connected planar domains and all other quasiconformal images of inner uniform domains. A technical assumption such as slice is needed for this equivalence in order to outlaw trivial counterexamples constructed via removability or extendability results for Sobolev spaces.

Hölder domains are not necessarily one-sided slice domains in the sense of [BuKo] (or in the sense defined in the previous section, for that matter). However, weak slice conditions give us a nice equivalence.

Theorem 3.4. *A domain $\Omega \subsetneq \mathbb{R}^n$ is a Hölder domain if and only if it is both a one-sided wslice domain and a Trudinger domain.*

This equivalence for bounded domains is a special case of [BuOS, Corollary 5.4]. But Hölder domains and Trudinger domains are both necessarily bounded (for the latter, see [Bu, Proposition 2.4]), so an explicit boundedness hypothesis is not needed.

Unlike the two-sided slice condition, one-sided (weak or strong) slice conditions are not quasiconformally invariant. Indeed, a quasiconformal image of a Hölder domain may fail to be a one-sided weak slice domain; see [Bu, Theorem 3.6]. However the so-called *k-cap condition*, defined in [Bu], is a quasiconformal invariant. This condition is related to, but strictly weaker than, the weak slice condition. It is however strong enough that the combination of k-cap plus Trudinger is equivalent to Hölder. It thus leads to the following theorem.

Theorem 3.5 [Bu, Theorem 2.1]**.** *If $f : \Omega \to f(\Omega)$ is a quasiconformal mapping and Ω is a Hölder domain, then $f(\Omega)$ is a Hölder domain if and only if it is a Trudinger domain.*

3.6. Applications to quasiconformal equivalence.

The following result is an improvement of the $\alpha = 0$ case of [BuSt2, Theorem 4.1]; see also [BoHeKo, Proposition 7.12].

Theorem 3.7. *The following are equivalent for a bounded domain $\Omega = U \times V$:*

(1) *Ω is a Gromov hyperbolic domain;*
(2) *Ω is an inner slice domain;*
(3) *Ω is a wslice domain;*
(4) *both U and V are inner uniform domains;*
(5) *Ω is an inner uniform domain.*

Note that Theorem 3.7 also tells us that the two-sided slice and inner wslice conditions are equivalent to inner uniformity in the case of bounded product-type domains, since those conditions are intermediate between two-sided wslice and inner slice conditions. Thus Gromov hyperbolicity and the various slice-type conditions are well-understood for bounded product domains. We postpone the proof of Theorem 3.7 until the next section, where we also consider the relations between the various slice conditions on general domains (see the implication diagram on page 11).

The following theorem is an easy corollary of Theorems 3.2 and 3.7.

Theorem 3.8. *A bounded domain $\Omega = U \times V$ is quasiconformally equivalent to an inner uniform domain if and only if it is an inner uniform domain.*

4. Other results

4.1. Proving that a weak slice condition fails.

To prove that a domain does not satisfy an (inner) weak slice-type condition, we need good upper bounds for $\mathrm{WS}(x, y; C)$ or $\mathrm{WS}_{\mathrm{in}}(x, y; C)$. Here we state two lemmas (4.2 and 4.4) that have proved useful in this regard, the first exploiting the Euclidean metric and the second the quasihyperbolic metric. Both lemmas are more abstract and general than the versions one might first come up with, but this extra generality has proved useful in applications. Our first lemma is essentially [BuSt3, Lemma 2.17] with $\alpha = 0$ and Euclidean measure.

Lemma 4.2. *Suppose that $\Omega \subsetneq \mathbb{R}^n$ is a domain, $A \subset \Omega$ is a rectifiable set, and $\mathcal{F}$ is a collection of disjoint non-empty bounded subsets of Ω satisfying the following conditions:*

(1) *for each $S \in \mathcal{F}$, $\mathrm{len}(S \cap A) \geq c_S > 0$;*
(2) *there exists $\epsilon > 0$ and a function $g : A \to [\epsilon, \infty)$ such that $c_S \geq g(z)$ whenever $S \in \mathcal{F}$ and $z \in S \cap A$;*
(3) *there exists $C_0 > 0$ such that $\mathrm{len}(g^{-1}(0, t]) \leq C_0 t$ for all $t > 0$.*

Then the cardinality of $\mathcal{F}$ is at most $2C_0 \log_2(4 \,\mathrm{len}(A)/\epsilon)$.

Proof. We partition $\mathcal{F}$ into subsets $\mathcal{F}_j$ defined by the equation

$$\mathcal{F}_j = \{S \in \mathcal{F} \mid c_S \in (2^{j-1}, 2^j]\}, \qquad j \in \mathbb{Z}.$$

The set $g^{-1}((0, 2^j])$ has length at most $C_0 2^j \wedge \mathrm{len}(A)$, and it contains $S \cap A$ for all $S \in \mathcal{F}_j$. But if $S \in \mathcal{F}_j$, then $\mathrm{len}(S \cap A) > 2^{j-1}$. It follows that the cardinality of each $\mathcal{F}_j$ is at most $2C_0$, and that $\mathcal{F}_j$ is empty if $2^j > 2 \,\mathrm{len}(A)$. Since $\mathcal{F}_j$ is also empty for $2^j < \epsilon$, the lemma follows easily. □

We now turn to the proof of Theorem 3.7. We use ideas from this last lemma and its proof, although we do not appeal directly to it. The $\alpha = 0$ case of [BuSt2, Theorem 4.1] says that for a bound product-type domain $\Omega = U \times V$, the inner uniformity of Ω is equivalent to that of U and V, and to Ω satisfying a so-called inner 0-wslice$^+$ condition, a condition formally stronger than the two-sided inner wslice condition. It is not hard to prove that an inner uniform domain is an inner slice domain (this is a special case of [BaBu, Theorem 4.6]), so to prove Theorem 3.7 we need only prove that if $\Omega = U \times V \subset \mathbb{R}^m \times \mathbb{R}^l$ is a bounded one-sided wslice domain, then U and V are inner uniform. By symmetry, we consider only U.

We pick a point $v_0 \in V$ which maximizes distance to the boundary in V, and write $\delta_0 = \delta_V(v_0)$. Suppose $u, u' \in U$ and let $z_0 = (u, v_0)$, $z_0' = (u', v_0)$. Let $\gamma : [0,1] \to \Omega$ be a quasihyperbolic segment from z_0 to z_0' in Ω. Clearly, the V-coordinate of γ is constant, its U-coordinate γ_U is a quasihyperbolic segment from u to u' in U, and $k_U(u,u') \le k_\Omega(z_0, z_0')$. Suppose $\mathcal{F}$ is a collection of C-wslices for the pair $z_0, z_0' \in \Omega$, with $d_S = \operatorname{dia}(S)$, $S \in \mathcal{F}$. We assume without loss of generality that $\delta_\Omega(z_0) \le \delta_\Omega(z_0')$ and that each $S \in \mathcal{F}$ satisfies (WS-5); this may require a controlled increase in the value of C. Since $\delta_\Omega(z_0) = \delta_U(u) \wedge \delta_0$, it follows that $\delta_\Omega(z_0) = \delta_U(u) \wedge \delta_U(u') \wedge \delta_0$.

We claim that each $S \in \mathcal{F}$ is contained in the ball $B(z, (2C+2)l(u,u'))$ for $z = z_0, z_0'$. Take a path $\lambda_U \in \Gamma_U(u,u')$ of length less than $2l(u,u')$. Defining λ to be the path with first coordinate given by λ_U and constant second coordinate v_0, we have a path λ of length less than $2l(u,u')$. If $S \in \mathcal{F}$ were not contained in $B(z, (2C+2)l(u,u'))$ for one or both of the points $z = z_0, z_0'$, then the fact that S intersects λ would imply that $d_S \ge 2Cl(u,u')$, and so S could not satisfy (WS-1). This finishes the proof of our claim.

We next pick a point $v_1 \in V$ such that $\delta_0/2 < |v_1 - v_0| < \delta_0$, and let $z_1 = (u, v_1)$, $z_1' = (u', v_1)$. Define a path $\nu : [0,3] \to \Omega$ as follows:

$$\nu(t) = \begin{cases} (1-t)z_0 + tz_1, & 0 \le t \le 1, \\ (\gamma_U(t-1), v_1), & 1 \le t \le 2, \\ (3-t)z_1' + (t-2)z_0', & 2 \le t \le 3. \end{cases}$$

For each $j \in \mathbb{N}$, let A_j be the annular regions given by

$$A_j = \left\{ z \in \Omega \;\middle|\; 2^{j-1} \le \frac{C(|z - z_0| \wedge |z - z_0'|)}{\delta(z_0)} < 2^j \right\}.$$

We partition $\mathcal{F}$ into subsets $\mathcal{F}_j$ consisting of all those $S \in \mathcal{F}$ which intersect $A_j \cap \gamma$ but are disjoint from $A_i \cap \gamma$ for all $i > j$. Since the distance from $A_j \cap \gamma$ to ν is at least $(\delta_0/2) \wedge |z - z_0| \wedge |z - z_0'|$ and every $S \in \mathcal{F}$ must also intersect ν, we have $2d_S \ge \delta_0 \wedge (2^j C^{-1} \delta(z_0))$ for all $S \in \mathcal{F}_j$. Thus each $S \in \mathcal{F}_j$ contains a ball $B(z_S, d_S/C)$ of radius comparable with $2^{j-1}\delta(z_0)$ and is contained in the set $B(z_0, 2^j\delta(z_0)/C) \cup B(z_0', 2^j\delta(z_0)/C)$. By disjointness of the slices, this gives an upper bound N on the cardinality of $\mathcal{F}_j$, where N depends only on $m + l$, C, and δ_0; in particular N is independent of j. Also $\mathcal{F}_j$ is empty whenever $2^{j-1}\delta(z_0)/C >$

$(2C{+}2)l(u,u')$. Thus the cardinality of $\mathcal{F}$ is at most $N\log_2(4(C^2{+}C)l(u,u')/\delta(z_0))$ and so

$$\begin{aligned} k_U(u,u') &\le C\left(1+N\log_2\left(\frac{4(C^2+C)l(u,u')}{\delta(z_0)}\right)\right) \\ &\le C\left(1+N\log_2\left(\frac{4\delta_0^{-1}(C^2+C)l(u,u')}{\delta_U(u)\wedge\delta_U(u')}\right)\right). \end{aligned}$$

According to [Va2; Theorem 2.33], this condition implies inner uniformity, thus finishing the proof of Theorem 3.7.

The $\alpha>0$ case of [BuSt2, Theorem 4.1] involves slice conditions that we have not discussed in this paper. However, let us note that that result can be improved, using an $\alpha>0$ variant of Lemma 4.2, to the statement that for bounded product domains, the α-wslice, inner α-wslice$^+$, and inner α-mCigar conditions are all equivalent.

The proof of the following result is just a one-sided analogue of the proof of Theorem 3.7.

Theorem 4.3. *The following are equivalent for a bounded domain $\Omega=U\times V$:*

(1) *Ω is a one-sided inner slice domain;*
(2) *Ω is a one-sided wslice domain;*
(3) *Ω is a Hölder domain.*

We now come to the second lemma, which is a special case of [BuSt3, Corollary 2.9].

Lemma 4.4. *Suppose $C\ge 2$, $x,y\in\Omega$, and $A\subset\Omega$. Suppose that $\mathcal{F}$ is a finite set of bounded pairwise disjoint subsets of Ω satisfying (WS-2), and for which* $\operatorname{len}(A\cap S)\ge C^{-1}\operatorname{dia}(S)$ *for all $S\in\mathcal{F}$. Furthermore, we assume that each $S\in\mathcal{F}$ intersects every $\lambda\in\Gamma(x,y)$. Then there exists $c=c(C)>0$ such that* $\operatorname{len}_k(A\cap S)\ge c$ *for each $S\in\mathcal{F}$. Consequently, the cardinality of $\mathcal{F}$ is at most* $\operatorname{len}_k(A)/c$.

This lemma provides a rather simple method of producing domains that do not satisfy α-wslice conditions. Suppose $G\subsetneq\mathbb{R}^n$ is a domain and that $\Omega=G\setminus E$ is a subdomain of G such that $E\cap K$ is a finite set for all compact subsets K of G. Suppose now that $(x,y,\mathcal{F})$ is a C-wslice dataset with respect to the domain Ω, and that $\delta(z)\approx\delta_G(z)$ for $z=x$ and $z=y$. Switching the domain to be G does not affect (WS-2), and it is a routine exercise to show that each $S\in\mathcal{F}$ also satisfies (WS-1) for x,y with respect to the domain G (hint: make little bypasses around each point of E that a path λ passes through). Consequently, Lemma 4.4 tells us that

$$\mathrm{WS}(x,y;C)\le C(1+k_G(x,y)),$$

where $\mathrm{WS}(x,y;C)$ is defined relative to the domain Ω. We can therefore produce a domain that fails to satisfy a wslice condition by removing a countable family E of points which force paths between the points x,y go through bottlenecks tight enough that $k_\Omega(x,y)$ is much larger than $k_G(x,y)$.

Example 4.5. Let $t_j \in (0,1/2)$ with $t_{j+1} \le t_j/2$ for each $j \in \mathbb{N}$. Let G be the planar triangle $\{(x,y) : 0 < x < 1, |y| < x\}$ and let $E = \bigcup_{j=1}^{\infty} E_j$, where E_j consists of the n_j points which divide the line segment $\{(t_j,y) : |y| < t_j\}$ into n_j+1 equal subsegments, where $n_j \in \mathbb{N}$. Then $\Omega \equiv G \setminus E$ is an (inner) wslice domain if and only if the sequence (n_j) is bounded.

The case of a bounded sequence is clear, since Ω is then an inner uniform domain and so satisfies all slice-type conditions, so to justify the last statement we need to prove that a wslice condition fails if (n_j) is unbounded. This fact is proven using the technique given before this example: writing $z_j = (3 \cdot 2^{-j-2}, 0)$, $j \in \mathbb{N}$, both $k_G(z_j, z_{j-1})$ and $\delta_G(z_j)/\delta(z_j)$ are uniformly bounded over all $j \in \mathbb{N}$, while $k_\Omega(z_j, z_{j-1})$ is very large whenever n_j is very large.

An alternative approach can be used to prove the essentially weaker result that a strong (inner) slice condition fails if (n_j) is unbounded. Consider the ball separation condition for the geodesic $\gamma = [z_j, z_{j-1}]$, where z_j is as in the previous paragraph. Picking the intermediate point $w = w_j \in \gamma$ so that the first coordinate of w is 2^{-j}, we see that a large constant C is required in (BS) if n_j is large. Thus Ω is not a ball separation domain, and so not an (inner) slice domain, if (n_j) is unbounded.

If we are interested in the question of when Ω is a one-sided wslice domain, we can appeal to Theorem 3.4. Since the set E is removable for Sobolev spaces (this follows from the ACL-characterization of Sobolev spaces [Mz, Section 1.1.3]), Ω is a Trudinger domain. Thus Ω is a one-sided wslice domain if and only if it is a Hölder domain. It is then a routine exercise to give a characterization in terms of the sequence (n_j) alone. In fact Ω is a one-sided wslice domain if and only if the sequence $\left(j^{-1}\sum_{i=1}^{j} \log n_j\right)$ is bounded.

Example 4.6. Another typical example of a domain that does not satisfy a slice condition is $\Omega = G \times (-1,1)$, where G is a cusped domain such as

$$\{(u_1,u') \in \mathbb{R} \times \mathbb{R}^m \,:\, 0 < u_1 < 2,\ |u'| < u_1^s\}$$

for some fixed $s > 1$. Letting $\gamma(t) = (t,0,0)$ for all $0 \le t \le 1$, it is easy to see that the restriction of γ to $[a,1]$, $0 < a < 1$, is the unique k-geodesic segment from $(a,0,0)$ to $(1,0,0)$, However, this geodesic segment does not uniformly satisfy a separation condition as a tends to zero. Alternatively, Ω is an bounded product domain but is clearly not a Hölder domain, so Theorem 4.3 implies that it is not even a one-sided wslice domain.

4.7. Proving that a weak slice condition holds.

Let us briefly discuss how to prove a weak slice condition using explicitly constructed weak slices. There are two such ways that occur repeatedly. Typically a bare-hands example involves considering several cases and in each case we use some variant of one or other of these two constructions. The first method, applicable to all domains, is to use concentric annuli with geometrically increasing radii $A_i = B_l(x, 2^i\delta(x)) \setminus B_l(x, 2^{i-1}\delta(x))$ for $1 \le i \le m_0$, where

$$m_0 = \left\lfloor \log_2\left(\frac{l(x,y)}{\delta(x) \wedge \delta(y)}\right)\right\rfloor - 1,$$

assuming $m_0 > 0$. By swapping x and y if necessary, this allows us to derive the general estimate

$$\mathrm{WS}_{\mathrm{in}}(x, y; C) \geq 1 \vee \log_2 \left(\frac{l(x,y)}{2(\delta(x) \wedge \delta(y))} \right) \tag{4.8}$$

whenever $C \geq 4$. With this estimate, it is straightforward to prove that inner uniform domains are inner wslice domains, and that Hölder domains are one-sided wslice domains.

The second method is applicable when a domain has a *long corridor*. Consider, for instance, the simple case of a rectangular box $R \subset \mathbb{R}^3$ with normals in the coordinate directions, which is of sidelength 1 in the second and third coordinates, and sidelength N in the first, where $N \in \mathbb{N}$. We chop R into N disjoint open cubes Q_i of sidelength 1 (plus $n-1$ connecting squares which we discard). If $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ both lie in R, with $x_1 \leq y_1$, we choose as our slices all those cubes Q_i whose points have first coordinates larger than $x_1 + 1$ and less than $y_1 - 1$. These *corridor slices* suffice to verify a wslice condition for the pair x, y unless either $\delta(x)$ or $\delta(y)$ is less than $2^{x_1 - y_1}$. In this case, a wslice condition follows by instead using concentric annular slices.

Example 4.9. Consider the three-dimensional "rooms and corridors" domain $\Omega = \bigcup_{j=0}^{\infty} (R_j \cup K_k)$. Each room R_j is a rather flat box of dimensions $2^{-j} \times 2^{-j} \times 4^{-j}$ connected along opposite (non-square) ends to the (square) ends of K_{j-1} and K_j, where the corridor K_j is a box of dimensions $2^{-j} \times 4^{-j} \times 4^{-j}$. Then we claim that Ω is a one-sided weak slice domain but not a one-sided (strong) slice domain.

Let us justify the above claim. Picking x_0 to be the center of R_0, a path γ from x_0 to a point in R_m for large m must pass through R_j and K_j for each $1 \leq j < m$. These intermediate rooms R_j are problematic for large j, since γ must pass through some point $x_j \in R_j$ whose distance from the nearest corridor is at least 2^{-j-1}. If S_i is a slice containing x_j, it follows from (Sli_1) that d_i, and hence $\mathrm{len}(\gamma \cap S_i)$, is bounded from below by $c2^{-j}$, and so $\mathrm{dia}_k(\gamma \cap S_i)$ is bounded from below by $c2^j$; here c depends only on the slice constant C. This is inconsistent with (Sli_2) when j is large.

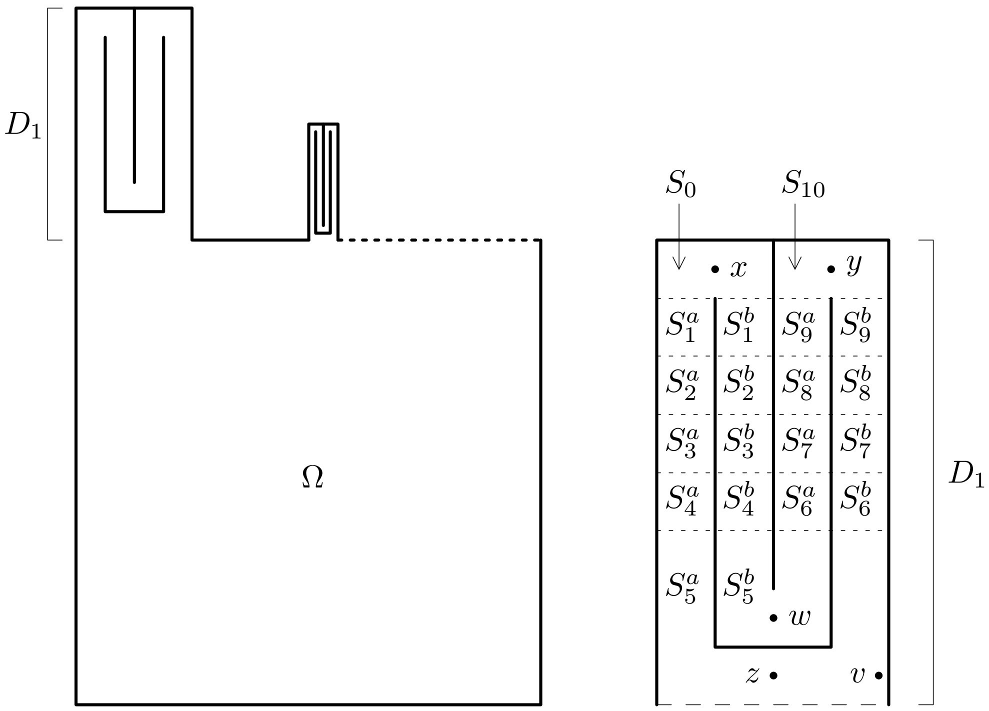
However, the rooms do not cause a problem for the wslice condition since we simply ignore them! We instead chop all intermediate corridors into corridor slices. In this way, we see that $\mathrm{WS}_{\mathrm{in}}(x_0, y; C) \geq 2^{m-1}$ whenever y is a point in $R_m \cup K_m$. This estimate is good enough to give a wslice condition unless $k(x, y)$ is much larger than 2^m, in which case we must have $\delta(x) < \exp(-2^m)$. In that case, concentric annular slices show that a wslice condition is satisfied.

Finally, let us discuss what is known and not known about how various slice-type conditions compare with each other. We have the following set of elementary implications among the various slice conditions; see [BuSt2, Section 5]. These implications hold whether we are discussing one-sided or two-sided conditions (as long as we do not mix one-sided and two-sided conditions).

$$\begin{array}{ccc} \text{Inner slice} & \Longrightarrow & \text{Inner wslice} \\ \Downarrow & & \Downarrow \\ \text{slice} & \Longrightarrow & \text{wslice} \end{array}$$

Examples are given in [BuSt2] to show that neither a one-sided slice condition nor a one-sided inner wslice condition implies the other one. Thus, in the case of one-sided conditions, all implications that do not logically follow from the above diagram are false.

The story for two-point conditions is less complete. One can construct a wslice domain that is neither a slice domain nor an inner wslice domain; see [BuDiSt] for details. Also one can construct a slice domain that is not an inner wslice domain; see Example 4.11 below. However, the question of whether or not an inner wslice domain is an inner slice domain is still open.



4.10. The domain Ω (left) and an annotated D_1 (right)

Example 4.11. The domain Ω above consists of a square of unit length to which we attach along one side a sequence of decorations D_j, $j \in \mathbb{N}$. The decoration D_j consists of a rectangle R_j of height 2^{-j} and width 4^{-j} from which we remove four line segments. It is convenient to coordinatize R_j with Euclidean coordinates that are shifted and rescaled by a factor 4^{j+1}, so that the corners of R_j have coordinates $(\pm 2, 0)$ and $(\pm 2, 2^{j+2})$. The first removed line segment is from $(0, 2^{j+2})$ to $(0, 2)$, while the other three form a U-shape from $(-1, 2^{j+2} - 1)$ down to $(-1, 1)$, then right to $(1, 1)$, and finally up to $(1, 2^{j+2} - 1)$. We claim that this domain is a slice domain but not an inner wslice domain.

The above domain Ω was earlier discussed in [BuSt3, Example 3.3], where it is shown to be a wslice domain that fails to be an inner wslice domain. Using that result, we need only prove that Ω is a slice domain to justify our claim. The proof that Ω is a slice domain is not unlike the proof that it is a wslice domain, but there are extra complications. Let us therefore begin by recapping why it is a wslice domain.

For each $j \in \mathbb{N}$, we slice D_j along horizontal dotted lines corresponding to integer values i of the second coordinate, $3 \le i \le 2^j - 1$. This is illustrated in the annotated close-up of D_1 in Figure 4.10; for now ignore the dashed line at the base of D_1 and the textual annotation. As always, proving a wslice condition involves a case analysis involving either corridor slices or concentric annular slices. Let us write $m(z) = j$ if $z \in D_j$. Concentric annuli always work for pairs x, y unless one or both points lie in a decoration (since the unit square is a uniform domain). If $m(x) = j$, we take as wslices all rectangular parts of D_j that are bounded above and below by dotted lines, which are an inner distance at least 2^{-2j-1} from both x and y, and which all paths from x to y must pass through. If $m(y) = i \ne j$, we also take as wslices all rectangular parts of D_i that are bounded above and below by dotted lines, which are an inner distance at least 2^{-2j-1} from y (and automatically from x), and which all paths from x to y must pass through (which in this case simply means that they lie below y). As the reader may check, there exists a universal constant C such that one or the other of these two sets of wslices gives a C-wslice condition for x, y whenever the concentric annular slice estimate (4.8) does not give a wslice condition for x, y. For instance for the pair x, y in Figure 4.10, there are four such wslices in D_1.

However, this method of constructing wslices does not allow one to prove a (strong) slice condition. The problem is that the quasihyperbolic diameters of such wslices are not uniformly bounded. To get a set of slices, we need to make two changes: we add a few extra slices to ensure that all of the quasihyperbolic geodesic is covered, and we typically divide each of the above wslices into two slices. A complication in this second change is that the manner of division is sensitive to the relative positions of our pair of points. As for the wslice proof, a proof involves a detailed case analysis which employs concentric annular slices, corridor slices or both in all cases. We leave this analysis to the reader, indicating the ideas by considering a few representative cases only.

Consider first the pair x, y in the Figure 4.10. Disjoint from our four previous wslices, we choose slices S_0, S_{10}, and S_5, the last of which consists of two pieces labelled S_5^a and S_5^b. We also split each of the wslices into two slices, giving us S_i, $1 \le i \le 9$, $i \ne 5$; like S_5, each of these slices has two pieces distinguished in the diagram by an a or b superscript. For the pair z, y, a subset of these slices suffices (namely those with indices $5 \le i \le 10$). For the pair v, y, where v is very close to the boundary, we would need to include some concentric annular slices near v which would then be excised from S_5.

So far, minor variations of one method of dividing wslices into slices has sufficed. But for a pair such as z, w in Figure 4.10, this is not a good method of defining slices, since the analogous construction for points in similar position in D_j for large j fails to uniformly satisfy (Sli$_2$). The problem is that $[z_1, w_1]$ passes through both parts of S_6 and these parts, being separated by two long corridors, are not close in the quasihyperbolic metric. We therefore reorganize our slices choosing S_5^a and S_5^b as the first and last slices, respectively, and choosing nine intermediate slices: $S_4^a \cup S_6^a$, $S_4^b \cup S_6^b$, $S_3^a \cup S_7^a$, $S_3^b \cup S_7^b$, $S_2^a \cup S_8^a$, $S_2^b \cup S_8^b$, $S_1^a \cup S_9^a$, $S_1^b \cup S_9^b$, and $S_0 \cup S_{10}$. One can prove a slice condition for all pairs of points in Ω by using concentric annular slices and/or corridor slices, the latter being produced by one or other of the two methods given above.

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