A note on the ring axioms

It is well known that if \((R, +, \cdot)\) is a ring with unity 1, then the commutative law of addition \(x + y = y + x\) for all \(x, y \in R\) is redundant; i.e. it can be deduced from the other ring axioms. In such a ring, every \(x \in R\) can be written as a product \(x = x \cdot 1\). In fact the stated result holds under this weaker condition.

**THEOREM.** Let \((R, +, \cdot)\) be a ring in which every element is a product; i.e. given \(r \in R\), there exist elements \(s, t \in R\) such that \(r = s \cdot t\). Then the commutative law of addition, \(x + y = y + x\) for all \(x, y \in R\), is redundant.

**PROOF.** Let \(x\) and \(y\) be arbitrary elements of \(R\) and let \(x = a \cdot d\) and \(y = b \cdot c\). Consider \((a + b) \cdot (c + d)\). Now, using the right distributive law followed by the left distributive law we have

\[
(a + b) \cdot (c + d) = a \cdot (c + d) + b \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d.
\]

On the other hand, using the left distributive law followed by the right distributive law, we have

\[
(a + b) \cdot (c + d) = (a + b) \cdot c + (a + b) \cdot d = a \cdot c + b \cdot c + a \cdot d + b \cdot d.
\]

Next, using the left and right cancellation laws in the group \((R, +)\) we see that \(a \cdot d + b \cdot c = b \cdot c + a \cdot d\) i.e. \(x + y = y + x\).

The following example shows that if we drop the condition that every element of \(R\) is a product, then the theorem is no longer valid.

**Example.** Let \((G, +)\) be a non-commutative group, written additively, with identity element 0. For all \(a, b \in G\), define \(a \cdot b = 0\), and thus there exists an element of \(G\) which is not a product. Also \((G, +, \cdot)\) has all the ring properties except commutativity of +.

The final example shows that the condition that every element is a product is weaker than the condition that the ring has a one-sided unity.

**Example.** Let \(R = \{0, a, b, c\}\) and let \(R_1 = (R, +, \cdot)\) and \(R_2 = (R, +, \cdot)\) be rings defined by

\[
\begin{array}{cccc}
+ & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0
\end{array}
\quad
\begin{array}{cccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & a & b & c \\
c & 0 & 0 & 0 & 0
\end{array}
\quad
\begin{array}{cccc}
* & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & 0 \\
b & 0 & b & b & 0 \\
c & 0 & c & c & 0
\end{array}
\]

Then it is clear that \((R_1, +, \cdot)\) has left unity but not right unity, whereas \(R_2\) has right unity but no left unity. So their direct sum \(R_1 \oplus R_2\) has no left unity or right unity. But \(R_1 \oplus R_2\) does have the product property, for if \((x, y) \in R_1 \oplus R_2\) then \((x, y) = (a, y)(x, a)\).

**STEPHEN BUCKLEY**
(graduate student)

*Department of Mathematics, University College, Cork*