

PTOLEMAIC SPACES AND CAT(0)

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ABSTRACT. We consider Ptolemy's inequality in a metric space setting. It is not hard to see that CAT(0) spaces satisfy this inequality. Although the converse is not true in full generality, we show that if our Ptolemaic space is either a Riemannian or Finsler manifold, then it must also be CAT(0). Ptolemy's inequality is closely related to inversions of metric spaces. We exploit this link to establish a new characterization of Euclidean space amongst all Riemannian manifolds.

1. INTRODUCTION

Inversion (or reflection) about the unit sphere is a bijection on $\mathbb{R}^n \setminus \{0\}$, so we can pull back Euclidean distance to get a new distance on $\mathbb{R}^n \setminus \{0\}$: $i_0(x, y) = |x - y|/|x||y|$. Inversion has been generalized to the setting of a metric space (X, d) in [8]: for fixed $p \in X$, define

$$(1) \quad i_p(x, y) = \frac{d(x, y)}{d(x, p)d(y, p)}, \quad x, y \in X_p,$$

where $X_p := X \setminus \{p\}$. In general this is not a metric on X_p , but a related function $d_p : X_p \times X_p \rightarrow [0, \infty)$ is defined and shown to be a metric which is subordinate to, but comparable with, i_p ; see [8, Lemma 3.2].

The definition of d_p is more complicated than that of i_p , so it is natural to ask when i_p itself is a metric for all $p \in X$. This reduces to deciding if i_p satisfies the triangle inequality, and is easily seen to be equivalent to *Ptolemy's inequality*

$$(2) \quad d(x, y)d(z, p) \leq d(x, z)d(p, y) + d(x, p)d(y, z), \quad x, y, z, p \in X.$$

A metric space satisfying this inequality is said to be *Ptolemaic* [17], [5], [6], [13], [14], [1], [10], [11].

It is not hard to see that CAT(0) spaces are Ptolemaic; see Section 3. As a corollary, the inversions i_p are metrics in every CAT(0) space. We also discuss implications in the converse direction in Section 3. The natural converse fails: there exists a geodesic Ptolemaic space that is not CAT(0); see the comments after Theorem 1.1 in [11]. Nonetheless, if more structure is imposed on the space under consideration, it is possible to give converse statements. We prove the following result in Section 4.

Theorem 1.1. *A complete Riemannian manifold M is Ptolemaic if and only if it is CAT(0), or equivalently a Hadamard manifold.*

Date: 3/10/2008.

The first two authors were supported by Enterprise Ireland and Science Foundation Ireland.

It has been brought to our attention that a local version of Theorem 1.1 was proven in the Ph.D. thesis of D.C. Kay. However, as that thesis is not easy to access, we include a complete proof of Theorem 1.1.

One might hope to get more examples of Ptolemaic spaces in the class of Finsler manifolds than in its subclass of Riemannian manifolds. The following result, proven in Section 5, says that this is not the case; see the thesis of D.C. Kay for a related result.

Theorem 1.2. *A Ptolemaic Finsler manifold is necessarily Riemannian.*

Finally, it seems natural to ask when the inversion metrics on a Riemannian manifold are themselves Riemannian. This question in fact allows us to characterize Euclidean space among all complete Riemannian manifolds, as revealed by the following result which is proved in Section 6. Recall that a Finsler manifold is a smooth manifold M together with a function $F : TM \rightarrow [0, \infty)$ which is smooth on the complement of the zero section of TM , and which restricts on each tangent space to give a *Minkowski norm*. (A Minkowski norm is a norm which satisfies a certain strong convexity condition; see [2] for details.)

Theorem 1.3. *Let M be a complete Riemannian manifold. Then the metric space $(M \setminus \{p\}, i_p)$ has the structure of a Riemannian manifold for all $p \in M$ if and only if M is Euclidean space.*

ACKNOWLEDGEMENT. The authors wish to thank Jane McDougall for her careful reading of this paper and for useful discussions. The authors are also indebted to the referee whose suggestions and simplifications of proofs have been very useful in producing the final version of the manuscript.

2. PRELIMINARIES

A *Riemannian manifold* M is a smooth manifold endowed with a Riemannian metric tensor g , that is, a choice of inner product on each tangent space $T_x M$ depending smoothly on $x \in M$.

A *Finsler space* is a smooth manifold M endowed with a Finsler metric $F : TM \rightarrow [0, \infty)$ which is C^∞ on $TM \setminus \{0\}$ and whose restrictions $F_q = F|_{T_q M}$ to the tangent spaces $T_q M$, $q \in M$, are Minkowski norms (for details see for instance [2]). The Finsler metric is introduced as in the Riemannian case: If $\gamma : [a, b] \rightarrow M$ is a Lipschitz continuous curve, then the function $t \mapsto F(\gamma'(t))$ is measurable and

$$L_F(\gamma) := \int_a^b F(\gamma'(t)) dt$$

is the length of γ . The metric is then given by

$$d(x, y) := \inf_{\gamma} L_F(\gamma), \quad x, y \in M,$$

where the infimum is taken over all Lipschitz continuous curves $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$. Recall that the exponential map $\exp_q : T_q M \rightarrow M$ is defined as follows. Given $u \in T_q M$, $\exp_q u$ is the point in M reached by a geodesic segment

starting at q in direction of u which has length $F_q(u)$, assuming that such a geodesic movement is possible in M (for further details see [2]).

Throughout the rest of this paper, (X, d) is a metric space and $X_p := X \setminus \{p\}$. X is said to be *proper* if all its closed balls are compact.

The *one-point extension* of X is defined via

$$\hat{X} := \begin{cases} X & \text{when } X \text{ is bounded,} \\ X \cup \{\infty\} & \text{when } X \text{ is unbounded;} \end{cases}$$

the open sets in \hat{X} include those in X together with complements (in \hat{X}) of closed balls (in X). Thus when X is a proper space, \hat{X} is simply its one-point compactification. Given a subspace $Z \subset X$, we write \hat{Z} and $\hat{\partial}Z$ to denote the closure and boundary of Z in \hat{X} ; e.g., $\hat{Z} = \bar{Z}$ when Z is bounded and $\hat{Z} = \bar{Z} \cup \{\infty\}$ when Z is unbounded.

Recall the metric d_p mentioned in the introduction. When (X, d) is unbounded, there is a unique point p' in the completion of (X_p, d_p) which corresponds to the point ∞ in \hat{X} . (Any unbounded sequence in (X_p, d) is a Cauchy sequence in (X_p, d_p) , and any two such sequences are equivalent.) Because of this phenomenon, we define

$$(\text{Inv}_p(X), d_p) := (\hat{X}_p, d_p) = (\hat{X} \setminus \{p\}, d_p)$$

and we call $(\text{Inv}_p(X), d_p)$ the *inversion* of (X, d) with respect to the base point p . For example, with this definition, $\text{Inv}_p(X)$ will be complete (or proper) whenever X is complete (or proper).

Note that i_p is a metric if and only if for all $x, y, z \in X_p$,

$$i_p(x, y) \leq i_p(x, z) + i_p(z, y).$$

Writing $i_p(x, y) = d(x, y)/d(x, p)d(y, p)$ and similarly for the other two terms above and then multiplying across by $d(x, p)d(y, p)d(z, p)$, we see that this last inequality becomes

$$d(x, y)d(z, p) \leq d(x, z)d(p, y) + d(x, p)d(y, z), \quad x, y, z \in X_p,$$

so, as mentioned before, i_p is a metric for all $p \in X$ if and only if X is Ptolemaic.

A classical result says that Ptolemy's inequality holds in the Euclidean plane, with equality if and only if the points x, y, z , and p lie on a circle in that order. In fact, this follows readily from the fact that i_p is the pullback of Euclidean distance under inversion in the unit sphere. Ptolemy's inequality has also been studied in more general settings than Euclidean space; see for instance [14], [1], [10]. Of particular interest to us is the result of Schoenberg [17] which says that a normed space is Ptolemaic if and only if its norm is induced by an inner product.

We do not distinguish notationally between paths and their images. Suppose $\gamma : I \rightarrow X$ is a path, where $I \subset \mathbb{R}$ is an interval. If γ contains points $x, y \in X$, then $\gamma[x, y]$ denotes any subpath of γ containing the points $x, y \in X$. This may not be unique: we insist only that different instances of such notation in the same proof refer to the same subpath.

If (X, d) is rectifiably connected, we define the *intrinsic metric associated with d* by

$$l(x, y) := \inf\{\text{len}(\gamma) : \gamma \text{ is a path in } X \text{ containing } x, y\}.$$

We say that (X, d) is a *length space* if $l = d$. A path γ of length $d(x, y)$ joining $x, y \in X$ is called a *geodesic segment*, and is often denoted $[x, y]$. We call (X, d) a *geodesic space* if all pairs of points can be joined by geodesic segments, i.e. the above infimum is always attained.

A *geodesic triangle* $T(x, y, z)$ is a collection of three points $x, y, z \in X$ together with a choice of geodesic segments $[x, y]$, $[x, z]$ and $[y, z]$. Given such a geodesic triangle $\Delta := T(x, y, z)$, a *comparison triangle* in this paper will always mean a geodesic triangle in the Euclidean plane, $T(\bar{x}, \bar{y}, \bar{z})$, such that corresponding distances coincide: $d(x, y) = |\bar{x} - \bar{y}|$, $d(y, z) = |\bar{y} - \bar{z}|$, $d(z, x) = |\bar{z} - \bar{x}|$. Note that to avoid cluttered notation, we do not specify the space in the notation $T(\cdot, \cdot, \cdot)$; the space will always be clear from the context.

A *triangle map* (in X) is a function f from a geodesic triangle $T(x, y, z)$ in X onto a comparison triangle $T(\bar{x}, \bar{y}, \bar{z})$, such that $f(x) = \bar{x}$, $f(y) = \bar{y}$, $f(z) = \bar{z}$, and the restriction of f to any one of the three sides is an isometry. Triangle maps always exist, and are unique up to an isometry of \mathbb{R}^2 . (X, d) is said to be a *CAT(0) space* if it is geodesic and $d(u, v) \leq |f(u) - f(v)|$ for all $u, v \in \Delta$ whenever $f : \Delta \rightarrow \bar{\Delta}$ is a triangle map. A geodesic triangle is *nondegenerate* if each of its sides is strictly shorter than the sum of the lengths of the other two sides.

An important class of CAT(0) spaces is given by *Hadamard manifolds*, i.e. complete simply connected Riemannian manifolds of nonpositive sectional curvature. Indeed a complete Riemannian manifold is CAT(0) if and only if it is a Hadamard manifold; see II.1.5, II.1A.6, and II.4.1 in [7].

If x, y, z are distinct points in X , then we define the angle $\theta := \angle_x(y, z)$ by

$$c^2 = a^2 + b^2 - 2ab \cos \theta,$$

where $a := d(x, y)$, $b := d(x, z)$, and $c := d(y, z)$. Note that θ is the Euclidean angle at \bar{x} whenever $T(\bar{x}, \bar{y}, \bar{z})$ is a comparison triangle for $T(x, y, z)$. If $\gamma : [0, a] \rightarrow X$ and $\gamma' : [0, \bar{a}] \rightarrow X$ are geodesic paths, parametrized by arclength and with $\gamma(0) = \gamma'(0)$, then we define the (*Alexandrov*) *angle* $\angle(\gamma, \gamma')$ by

$$\angle(\gamma, \gamma') := \limsup_{t, t' \rightarrow 0} \angle_{c(0)}(c(t), c(t')).$$

For much more on CAT(0) spaces and comparison angles, we refer the reader to [7]. Let us note only that Alexandrov angles correspond to the usual definition of angles in Euclidean and hyperbolic space; for the hyperbolic case see [7, I.2.9].

3. CAT(0) SPACES ARE PTOLEMAIC

In this section, we discuss the following basic result.

Proposition 3.1. *Every CAT(0) space is Ptolemaic.*

This result follows easily from the fact (mentioned in [7, II.1.10]) that CAT(0) spaces (X, d) satisfy the following *CAT(0) four-point condition*:

For every choice of 4 points $x_1, x_2, y_1, y_2 \in X$, there exist points $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2 \in \mathbb{R}^2$ such that $d(x_i, y_i) = |\bar{x}_i - \bar{y}_i|$, $i, j = 1, 2$, and $d(x_1, x_2) \leq |\bar{x}_1 - \bar{x}_2|$ and $d(y_1, y_2) \leq |\bar{y}_1 - \bar{y}_2|$.

In fact, we simply take $(x_1, x_2, y_1, y_2) = (x, y, z, p)$ and use the fact that the Euclidean plane is Ptolemaic.

For the sake of completeness and since we shall later use some of the ideas contained in it, we also give the following direct proof:

A direct proof of Proposition 3.1. Let x, y, z and p be arbitrary points in X . Consider Euclidean comparison triangles with vertices $\bar{x}, \bar{y}, \bar{z}$ and $\bar{x}, \bar{y}, \bar{p}$ corresponding to the triangles in X with vertices x, y, z and x, y, p , respectively. We choose the comparison triangles in such a way that \bar{z} and \bar{p} are not in the same open half-plane with respect to the line through \bar{x} and \bar{y} .

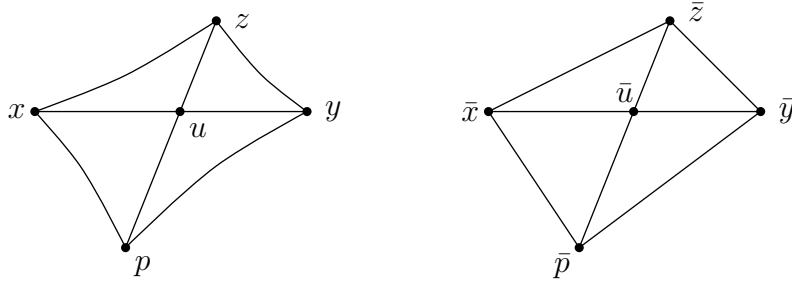


FIGURE 1. The case where $[\bar{z}, \bar{p}]$ intersects $[\bar{x}, \bar{y}]$

We distinguish between two cases. In the first case, illustrated in Figure 1, the closed line segment with endpoints \bar{z} and \bar{p} intersects the closed line segment with endpoints \bar{x} and \bar{y} . Let \bar{u} be the intersection point and let u be the corresponding point on the geodesic segment connecting x and y in X , that is, the unique point u on this geodesic segment such that $d(x, u) = |\bar{x} - \bar{u}|$. By the CAT(0) condition, we have $d(z, u) \leq |\bar{z} - \bar{u}|$ and $d(p, u) \leq |\bar{p} - \bar{u}|$, and so

$$(3) \quad d(z, p) \leq d(z, u) + d(u, p) \leq |\bar{z} - \bar{u}| + |\bar{u} - \bar{p}| = |\bar{z} - \bar{p}|.$$

It follows that Ptolemy's inequality holds for d , since all other distances involved are preserved, and the Euclidean plane is Ptolemaic.

We now consider the alternative Case 2, when the closed segment with endpoints \bar{z} and \bar{p} does not intersect the closed segment from \bar{x} to \bar{y} . In this case, we prove the stronger inequality

$$d(x, y)(d(z, x) + d(x, p)) \leq d(x, z)d(y, p) + d(x, p)d(y, z),$$

or equivalently

$$(4) \quad |\bar{x} - \bar{y}|(|\bar{z} - \bar{x}| + |\bar{x} - \bar{p}|) \leq |\bar{x} - \bar{z}||\bar{y} - \bar{p}| + |\bar{x} - \bar{p}||\bar{y} - \bar{z}|,$$

since the distances involved are all preserved by moving to the Euclidean plane.

To prove (4), we swivel the points \bar{z} and \bar{p} in such a way that the distances $|\bar{z} - \bar{x}|$ and $|\bar{x} - \bar{p}|$ are held constant and the angles $\angle_{\bar{x}}(\bar{z}, \bar{y})$ and $\angle_{\bar{x}}(\bar{p}, \bar{y})$ are decreasing. As long as the angles $\angle_{\bar{y}}(\bar{z}, \bar{x})$ and $\angle_{\bar{y}}(\bar{p}, \bar{x})$ are not allowed to exceed $\pi/2$, swiveling in this manner decreases the distances $|\bar{z} - \bar{y}|$ and $|\bar{p} - \bar{y}|$ while holding the three other distances in (4) constant; see Figure 2. Thus the left-hand side of (4) is invariant under

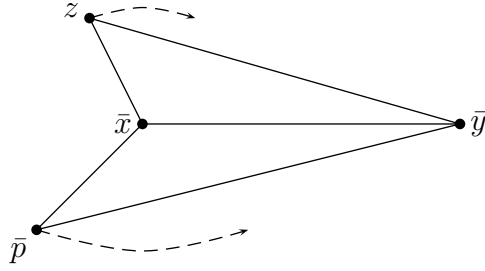


FIGURE 2. The nonintersecting case

such a swivel and the right-hand side decreases, so (4) holds prior to such a swivel if it holds after the swivel.

A moment's reflection shows that we can always perform a swivel of this type to get $\angle_{\bar{x}}(\bar{z}, \bar{p}) = \pi$ without allowing either $\angle_{\bar{y}}(\bar{z}, \bar{x})$ or $\angle_{\bar{y}}(\bar{p}, \bar{x})$ to exceed $\pi/2$. But then we are back in Case 1, and (4) coincides with the usual Ptolemaic inequality because $|\bar{z} - \bar{x}| + |\bar{x} - \bar{p}| = |\bar{z} - \bar{p}|$. This concludes the proof of (4) in Case 2, and the proof of the proposition. \square

Let us note that (4) fails for certain configurations in Case 1. In fact if we take $\bar{x}, \bar{z}, \bar{y}, \bar{p}$ to be four distinct points lying on a Euclidean circle in that order, then Ptolemy's theorem tells us that

$$|\bar{x} - \bar{y}| |\bar{z} - \bar{p}| = |\bar{x} - \bar{z}| |\bar{y} - \bar{p}| + |\bar{x} - \bar{p}| |\bar{y} - \bar{z}|.$$

Since $|\bar{z} - \bar{p}| < |\bar{z} - \bar{x}| + |\bar{x} - \bar{p}|$, we cannot have equality in (4).

A straightforward computation shows that Ptolemy's inequality (2) implies the triangle inequality for the inversion i_p defined in (1), thus proving the following corollary of Proposition 3.1.

Corollary 3.2. *If (X, d) is CAT(0), then i_p is a metric for all $p \in X$.*

Let us now discuss possible converses to Proposition 3.1. Since a Ptolemaic space need not be geodesic (for instance the set of integers with the Euclidean metric inherits the Ptolemaic property from \mathbb{R}), one could not hope for all Ptolemaic spaces to be CAT(0). But it is not even true that Ptolemaic spaces can always be isometrically imbedded in a CAT(0) space. Consider for instance the space

$$X = \{(0, 0), (0, 1), (1, 1), (1, 2)\}$$

with l^∞ metric

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

If (X, d) is a subspace of a geodesic space (Y, d) , then the concatenation of geodesic segments $[(0, 0), (0, 1)]$ and $[(0, 1), (1, 2)]$ gives one geodesic segment from $(0, 0)$ to $(1, 2)$ while the concatenation of geodesic segments $[(0, 0), (1, 1)]$ and $[(1, 1), (1, 2)]$ gives another. The two segments have to be different since any segment from $(0, 0)$ to $(1, 2)$ that goes through both $(0, 1)$ to $(1, 1)$ must have length at least 3. The CAT(0) condition readily implies uniqueness of geodesic segments so we are done.

A weaker converse implication would be the statement that a geodesic Ptolemaic space is CAT(0). This is not true in general (see [11]), but we prove it in the context of Riemannian manifolds in the next section.

4. PTOLEMAIC RIEMANNIAN MANIFOLDS ARE CAT(0)

In this section we establish a converse to Proposition 3.1 in the Riemannian setting: a complete Riemannian manifold (M, g) whose metric satisfies Ptolemy's inequality is necessarily CAT(0). In view of the fact that a CAT(0) manifold is precisely a Hadamard manifold, we must prove that a complete Ptolemaic manifold is simply connected (a global property) and of nonpositive sectional curvature (a local property). It makes sense to separate these two implications. We begin with the global one, which we prove in a more general context than manifolds.

Theorem 4.1. *If a Ptolemaic length space is complete and semilocally simply connected, then it must be simply connected.*

Before giving the proof, let us explain the idea. First note that a circle of perimeter $4L$ with the arclength metric attached is not Ptolemaic. Indeed, if we choose the points x, z, y, p to be *quarter points* on the circle, i.e. four points in this order around the circle that divide the circle into four pieces of equal length, then

$$d(x, y)d(z, p) = (2L)(2L) = 4L^2 > L^2 + L^2 = d(x, z)d(y, p) + d(x, p)d(y, z).$$

Thus any metric space X containing an isometric copy of a circle is non-Ptolemaic. A space that fails to be simply connected might fail to have an isometric copy of a circle, but Ptolemy's inequality still fails if we can find a set of four points in X for which the set of distances between pairs approximates sufficiently well the distances between quarter points of a circle. The proof of Theorem 4.1 amounts to finding a set of such approximate quarter points.

Recall that a topological space X is *semilocally simply connected* if each point $x \in X$ has a neighborhood U such that each loop at x contained in U is contractible within X .

Proof of Theorem 4.1. We assume that (X, d) is a complete semilocally simply connected length space which fails to be simply connected. To prove the theorem, we are required to prove that X cannot be Ptolemaic. The proof is a stopping time construction involving noncontractible loops γ_j at basepoints p_j , for $j \leq j_0$ where $j_0 \in \mathbb{N} \cup \{\infty\}$ is the stopping index.

We begin by picking any basepoint $p_1 \in X$ and a noncontractible loop $\gamma_1 : [0, 4L_1] \rightarrow X$ parametrized by arclength with $\gamma_1(0) = \gamma_1(4L_1) = p_1$ and the following near-minimality property: $4L_1 < (7/6) \text{len}(\gamma)$ for all noncontractible loops γ at p_1 . Note that semilocal simple connectedness guarantees that this is possible. By writing $l_1 := \gamma_1(L_1)$, $u_1 := \gamma_1(2L_1)$, $r_1 := \gamma_1(3L_1)$, we divide the loop into four subpaths of length L_1 .

It is convenient to define the *left half* and *right half* of γ_1 to be the paths from p_1 to u_1 given by the joins $\gamma_1[p_1, l_1] + \gamma_1[l_1, u_1]$, and $\gamma_1[u_1, r_1] + \gamma_1[r_1, p_1]$, respectively. Similarly, we define the *upper* and *lower halves* of γ_1 to be the paths from l_1 to r_1 obtained by joining the two of these four subarcs of γ_1 that have u_1 or p_1 as an endpoint, respectively.

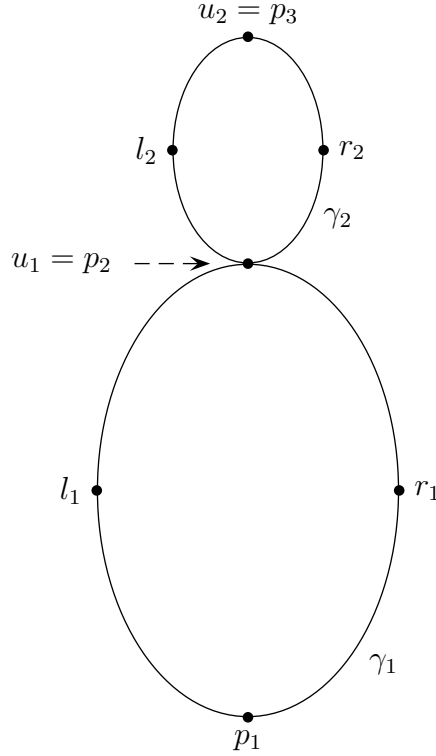


FIGURE 3. The first two steps of the construction

We claim that $d(p_1, u_1) \geq 10L_1/7$. Suppose for the sake of contradiction that this is not true, and so we can find a path λ_1 from p_1 to u_1 of length less than $10L_1/7$. Joining λ_1 with either the left or right half of γ_1 , we get two loops at p_1 . Since γ_1 is noncontractible, at least one of these two new loops is noncontractible. But both have length less than $2L_1 + 10L_1/7 = (6/7)(4L_1)$ and this contradicts the assumed near-minimality of γ_1 . Thus the claim follows.

The stopping condition for this first step is $d(l_1, r_1) \geq 10L_1/7$. Since the distances $d(p_1, l_1)$, $d(l_1, u_1)$, $d(u_1, r_1)$, and $d(r_1, p_1)$ are all at most L_1 , and $d(p_1, u_1) \geq 10L_1/7$, it follows that if the stopping condition holds, then

$$d(l_1, r_1)d(p_1, u_1) \geq \left(\frac{10L_1}{7}\right)^2 > L_1^2 + L_1^2 \geq d(l_1, p_1)d(r_1, u_1) + d(l_1, u_1)d(r_1, p_1),$$

and so the space is not Ptolemaic, as required.

Suppose instead that the stopping condition fails, so that there exists a path ν_1 from l_1 to r_1 of length less than $10L_1/7$. Joining ν_1 with either the upper or lower half of γ_1 , we get two new loops which we term the *upper* and *lower loop*, respectively. The noncontractibility of the lower loop contradicts the assumed near-minimality of γ_1 as before, so we may assume that it is contractible and thus that the upper loop is noncontractible. Note that the noncontractibility of the upper loop does not lead to a contradiction because this loop does not pass through p_1 .

We define $p_2 = u_1$. Since the upper loop is noncontractible and has length less than $(6/7)4L_1$, semilocal simple connectedness implies that we can find a noncontractible loop $\gamma_2 : [0, 4L_2] \rightarrow X$, parametrized by arclength, with $\gamma_2(0) = \gamma_2(4L_2) = p_2$, $L_2 < (6/7)L_1$, and the following near-minimality property: $4L_2 < (7/6)\text{len}(\gamma)$ for all noncontractible loops γ at p_2 .

The construction proceeds in the obvious inductive manner. If at any stage the corresponding stopping condition applies, then the space is not Ptolemaic. To finish the proof, we will show that failure of the stopping condition at every step leads to a contradiction. Indeed this is easy: since the paths γ_j have length

$$L_j < (6/7)^{j-1}L_1 \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

the sequence (p_j) necessarily converges to some point $p \in X$ and, for sufficiently large j , the noncontractible path γ_j lies in an arbitrary small neighborhood of p , contradicting semilocal simple connectedness. \square

Let us remark that it is proved in [11, Theorem 1.2] that a proper Ptolemaic space is uniquely geodesic and so contractible. In particular it is simply connected. However properness is a rather strong condition, not satisfied by infinite dimensional normed spaces, for instance.

It remains to prove that a Ptolemaic Riemannian manifold has nonpositive curvature.

Theorem 4.2. *If a Riemannian manifold (M, g) of dimension $n \geq 2$ is Ptolemaic, then it has nonpositive sectional curvature.*

The proof of Theorem 4.2 requires a strict Topogonov type result comparing hinges in spaces of constant curvature, given in the following lemma, and a consequence of this lemma.

In the following two results, a *hinge* simply means a triple of points in a metric space. A pair of hinges $\{u, v, w\}$ and $\{\bar{u}, \bar{v}, \bar{w}\}$ (in typically different metric spaces both of whose metrics we denote by d) are said to be *comparison hinges* if $d(u, v) = d(\bar{u}, \bar{v})$, $d(u, w) = d(\bar{u}, \bar{w})$, and $\angle_u(v, w) = \angle_{\bar{u}}(\bar{v}, \bar{w})$. Note that the major difference between this concept and that of comparison triangles is that we require two distances and an angle to match rather than three distances.

Lemma 4.3. *Suppose we have comparison hinges $\{v, p_1, p_2\}$ in a space form of constant curvature $\delta \neq 0$ and $\{\bar{v}, \bar{p}_1, \bar{p}_2\}$ in \mathbb{R}^n . If the angle $\alpha = \angle_v(p_1, p_2) = \angle_{\bar{v}}(\bar{p}_1, \bar{p}_2)$ lies in $(0, \pi)$, then*

- (i) $d(p_1, p_2) < |\bar{p}_1 - \bar{p}_2|$ if $\delta > 0$;
- (ii) $d(p_1, p_2) > |\bar{p}_1 - \bar{p}_2|$ if $\delta < 0$.

Proof. Given the hinge in \mathbb{R}^n , we can assume the comparison hinge in the space form M arises in the following way. Choose any point $v \in M$, and identify the origin of the tangent space to M at this point with \bar{v} . Corresponding to the points $\bar{p}_1, \bar{p}_2 \in T_v M$, we set $p_i = \exp(\bar{p}_i)$ for $i = 1, 2$. This gives the desired hinge in M .

We focus first on the case $\delta > 0$. Consider the geodesic $\gamma(t)$ in $T_v M$ joining \bar{p}_1 to \bar{p}_2 . As $\alpha \neq 0, \pi$, for each t we have a non-zero component of $\gamma'(t)$ orthogonal to the line from \bar{v} to $\gamma(t)$. Since $|\bar{v} - \gamma(t)| > 0$, it follows easily from a result of Rauch [3, Proposition

74] that each of these orthogonal components is strictly longer than the image of the component under the (derivative of the) exponential map at \bar{v} . Since \exp_* preserves the length of vectors along any geodesic from \bar{v} , we see that

$$|\exp_* \gamma'(t)| < |\gamma'(t)|.$$

(For future reference, observe that if $\delta < 0$, this inequality would be reversed.) The path $\exp \gamma(t)$ joining p_1 to p_2 is therefore strictly shorter than $\gamma(t)$. But the path $\exp \gamma(t)$ has length at least as long as the distance in M between p_1 and p_2 . Hence $d(p_1, p_2) < |\bar{p}_1 - \bar{p}_2|$, as claimed.

We now consider the case $\delta < 0$. This time let $\gamma(t)$ be a minimal geodesic joining p_1 to p_2 in M . By the arguments used above, we see that the the path $\exp^{-1} \gamma(t)$ joining \bar{p}_1 to \bar{p}_2 in $T_v M$ has strictly shorter length than $\gamma(t)$, since the velocity vectors to $\exp^{-1} \gamma(t)$ must have some component orthogonal to the radial direction from \bar{v} , at least for t in some open interval. Moreover, this path in $T_v M$ is at least as long as the Euclidean distance between \bar{p}_1 and \bar{p}_2 . Hence $d(p_1, p_2) > |\bar{p}_1 - \bar{p}_2|$, as required. \square

Corollary 4.4. *Consider a hinge $\{w, q_1, q_2\}$ in a manifold M of curvature $K > 0$, with angle $\angle_w(q_1, q_2) \in (0, \pi)$, and let $\{\bar{v}, \bar{p}_1, \bar{p}_2\}$ be a comparison hinge in Euclidean space. Then $d_M(q_1, q_2) < |\bar{p}_1 - \bar{p}_2|$.*

Proof. The hinge $\{w, q_1, q_2\}$ lies inside some compact region of the manifold. Over this region, the curvature is bounded away from 0, that is, $K \geq \delta > 0$. Let $\{v, p_1, p_2\}$ be a comparison hinge in the space form X of constant curvature δ . By the Toponogov Theorem, we have $d_M(q_1, q_2) \leq d_X(p_1, p_2)$. By Lemma 4.3, we also have $d_X(p_1, p_2) < |\bar{p}_1 - \bar{p}_2|$. Thus $d_M(q_1, q_2) < |\bar{p}_1 - \bar{p}_2|$, as claimed. \square

Proof of Theorem 4.2. If the curvature of M fails to be non-positive, we can find $p \in M$ and a two dimensional subspace V of $T_p M$ such that the sectional curvature $K(V) > 0$. Restricting the exponential map at p to V , we obtain a local diffeomorphism from a neighbourhood of the origin in V onto a two-dimensional submanifold of M . By openness of the positivity condition, the Gaussian curvature of this submanifold will be strictly positive close to p . In particular there exists $\epsilon > 0$ such that the closed ϵ -ball centred at p in our submanifold consists entirely of points with positive Gaussian curvature. Call this ball B .

Now choose linearly independent vectors v_1, v_2 in V of equal length $< \epsilon$. Set $v_{-i} = -v_i$ for $i = 1, 2$; set $x_i = \exp(v_i)$ for $i = -2, -1, 1, 2$. Let d_B respectively d_M denote distances measured in B and M , and let d denote distances in V . Clearly $d_B(x_i, x_j) \geq d_M(x_i, x_j)$ in general, and $d_B(x_i, x_{-i}) = d_M(x_i, x_{-i}) = d(v_i, v_{-i})$. Linear independence means that for $|i| \neq |j|$, the angle $\angle_0(v_i, v_j) = \angle_p(x_i, x_j) \in (0, \pi)$, and since the curvature is strictly positive, Corollary 4.4 applies. From this we deduce that for such i, j ,

$$d_M(x_i, x_j) \leq d_B(x_i, x_j) < d(v_i, v_j).$$

Since the $\{v_i\}$ lie on a Euclidean circle, the Ptolemy inequality is in fact an equality for these points. However, the above estimates show that the Ptolemy inequality does not hold in M for the points $\{x_i\}$, so M cannot be a Ptolemaic space. \square

5. PTOLEMAIC FINSLER SPACES ARE RIEMANNIAN

In view of the afore-mentioned result of Schoenberg [17] that normed spaces are Ptolemaic if and only if their norm is induced by an inner product, one can also consider the question of finding a converse to Proposition 3.1 in the setting of Finsler geometry. As stated in Theorem 1.2, it turns out that if the Finsler metric of a Finsler space satisfies Ptolemy's inequality, then it is in fact Riemannian. The proof uses Schoenberg's result [17] and a theorem of Whitehead asserting that the differential at the origin of the exponential map of a Finsler space is always the identity [18], [2].

Proof of Theorem 1.2. There is a concept of exponential map for Finsler manifolds, and this map exists at least in some neighbourhood of each point; see [2, page 126]. Moreover, it is C^1 at the zero-section of TM , and smooth away from the zero-section [2, pages 126–127]. A theorem of Whitehead states that the derivative of the exponential map at its origin is always the identity [18], [2].

Given any point p in our Finsler manifold $(M^n; F)$, the existence of a pointed differentiable map $(T_pM, 0) \rightarrow (M, p)$ which is isometric at 0, such as the exponential map, is sufficient to ensure the pointed Gromov-Hausdorff convergence of the scaled spaces $(M, p; \lambda F)$ to the space $(T_pM, 0; F_p)$ as $\lambda \rightarrow \infty$. Here, F_p denotes the Minkowski norm on the tangent space T_pM . (See for example Proposition 3.15 of [15]. This deals with the Riemannian analogue of the above, but the same argument is easily seen to work in the Finsler case.)

The Ptolemaic condition is preserved both by scaling and under Gromov-Hausdorff convergence. Its preservation by scaling is trivial. To see why it is preserved under Gromov-Hausdorff convergence, consider a sequence of Ptolemaic spaces X_i converging to a metric space X . If X is *not* Ptolemaic, then we can find four points x, y, z, p such that

$$d(x, y)d(z, p) > d(x, z)d(y, p) + d(x, p)d(y, z).$$

Since this inequality is strict, there is an $\epsilon > 0$ such that the inequality

$$(d(x, y) - \epsilon)(d(z, p) - \epsilon) > (d(x, z) + \epsilon)(d(y, p) + \epsilon) + (d(x, p) + \epsilon)(d(y, z) + \epsilon)$$

holds. For this ϵ and for i sufficiently large, we can find ϵ -Hausdorff approximations $X_i \rightarrow X$ and $X \rightarrow X_i$ (see [12] page 146 for details). It follows from the second inequality above that the images in X_i of x, y, z, p under such a map must also fail to satisfy the Ptolemy inequality. Therefore, since each X_i is assumed Ptolemaic we deduce that we *cannot* find four points in X for which the Ptolemy condition fails. In other words, X must be Ptolemaic.

Given that our Finsler manifold is Ptolemaic, we now deduce that the tangent cone at p , $(T_pM; F_p)$ is also Ptolemaic. By a result of Schoenberg [17], this means that the norm F_p is induced by an inner product. Since this is true for all $p \in M$, we see that M is actually a Riemannian manifold, as claimed. □

Summarizing our results for manifolds so far, we have in effect proven the following equivalence.

Theorem 5.1. *For a smooth manifold M the following conditions are equivalent:*

- (i) M is Finsler and Ptolemaic.
- (ii) M is Finsler and $CAT(0)$.
- (iii) M is Hadamard.

Proof. The implication (ii) \Rightarrow (i) follows from Proposition 3.1 which states that Ptolemy's inequality is satisfied in any geodesic $CAT(0)$ -space. Note that this result holds in more generality than the Finsler manifolds we consider here.

In order to prove (i) \Rightarrow (iii) we first observe that by Theorem 1.2 the fact that Ptolemy's inequality is satisfied in the Finsler manifold M implies that M must be Riemannian. Furthermore, Theorem 4.1 ensures that M must be simply connected. Finally, if M is assumed not to have non-positive sectional curvature, it follows by Theorem 4.2 that Ptolemy's inequality fails. Therefore, M must have non-positive curvature, and thus is Hadamard.

The implication (iii) \Rightarrow (ii) is trivial since M is assumed to be Hadamard and is thus Finsler, and since any simply connected Riemannian manifold with non-positive curvature is $CAT(0)$. \square

6. WHEN ARE THE INVERSIONS OF A MANIFOLD RIEMANNIAN?

In this section, we prove Theorem 1.3. In fact we prove more, as revealed by the following result. (For the statement, recall that the only difference between (M_p, i_p) and $\text{Inv}_p(M)$ is that the latter space includes a point at infinity if M is unbounded.)

Theorem 6.1. *Let M be a complete Riemannian manifold of dimension $n \geq 1$. The following are equivalent:*

- (a) M is Euclidean n -space;
- (b) $\text{Inv}_p(M)$ has the structure of a Riemannian manifold, for all $p \in M$;
- (c) (M_p, i_p) has the structure of a Riemannian manifold, for all $p \in M$;
- (d) $\text{Inv}_p(M)$ is a length space, for all $p \in M$;
- (e) (M_p, i_p) is a length space, for all $p \in M$ ($n > 1$ only).

Proof. Most of the implications of this theorem are easy, so let us deal with the easy cases first. If M is Euclidean n -space, then we know that (M_p, i_p) is isometric to Euclidean n -space with one point removed and that $\text{Inv}_p(M)$ is isometric to Euclidean n -space, so it is immediate that (a) implies (b), (c), (d), and (e). Trivially (b) implies (c) and (d), both of which in turn imply (e). Moreover the case $n = 1$ is easy since the only complete Riemannian manifolds are $M = \mathbb{R}^1$ (in which case $\text{Inv}_p(M)$ is isometric to the Euclidean line and (M_p, i_p) is isometric to the Euclidean line with a point removed), and $M = \mathbb{S}^1$ (in which case i_p is not a metric by Theorem 4.1). Therefore it suffices to prove that (e) \Rightarrow (a), and we may assume that $n > 1$.

Note that by Theorem 1.1, (e) already implies that M is a Hadamard manifold. So all we need to show is that if the sectional curvature of M is nonzero at some point $p \in M$, then i_p is not a length metric. This follows from a more general statement proven in the next proposition. \square

Proposition 6.2. *Let X be a CAT(0) space and $p \in X$ such that i_p is a geodesic metric. Then, for all $x, y \in X$ such that x, y, p do not lie on a common geodesic, the triangle with vertices x, p, y is flat. Thus, X is isometric to a convex subset of a Euclidean cone with vertex p .*

Proof. Since x, y, p do not lie on a common geodesic, the geodesic triangle $\Delta := T(x, y, p)$ is nondegenerate. Suppose that there exists an i_p -geodesic segment γ in $\text{Inv}_p(M)$ for the pair x, y . In (X, d) , γ is either a single path between x and y (if $p' \notin \gamma$), or a pair of paths from x and y to infinity (if $p' \in \gamma$). In either case, we can pick a point $z \in \gamma$, with $0 < d(x, z) < d(x, y)/2$. Now $i_p(x, y) = i_p(x, z) + i_p(z, y)$, and it follows easily that

$$(5) \quad d(x, y)d(z, p) = d(x, z)d(p, y) + d(x, p)d(y, z).$$

We thus have equality in the Ptolemy inequality for the points p, x, y, z .

Recall that the subembedding theorem [7, II.1.10] states that CAT(0) spaces (X, d) satisfy the CAT(0) four-point condition: For every choice of 4 points $x_1, x_2, y_1, y_2 \in X$, there exist points $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2 \in \mathbb{R}^2$ such that $d(x_i, y_i) = |\bar{x}_i - \bar{y}_i|$, $i, j = 1, 2$, and $d(x_1, x_2) \leq |\bar{x}_1 - \bar{x}_2|$ and $d(y_1, y_2) \leq |\bar{y}_1 - \bar{y}_2|$. The rigidity part of the subembedding theorem states that if there are four points x_1, x_2, y_1, y_2 in a CAT(0) space X such that the above equalities hold, and the inequalities become equalities, then the convex hull of x_1, x_2, y_1, y_2 in X is isometric to the corresponding Euclidean comparison quadrangle. This statement can be proven along the following lines. Assume for simplicity that the points $\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2$ form a convex quadrilateral in \mathbb{R}^2 and let \bar{m} be the intersection point of the diagonals $[\bar{x}_1, \bar{x}_2]$ and $[\bar{y}_1, \bar{y}_2]$. Furthermore, let m_1 and m_2 be the comparison points corresponding to \bar{m} on the geodesic segments with endpoints x_1, x_2 and y_1, y_2 , respectively. By using the CAT(0) property and the triangle inequality one can readily show that m_1 and m_2 coincide. We can then apply [9, Proposition 9.1.19] to obtain that all triangles $T(x_1, y_1, x_2)$, $T(x_2, y_2, x_1)$, $T(x_1, y_1, y_2)$ and $T(y_1, x_2, y_2)$ are flat.

Now set $x_1 = x$, $y_1 = p$, $x_2 = y$, $y_2 = z$ and assume that either one or both the inequalities in the four-point condition is a strict inequality. Then, by (5),

$$\begin{aligned} |\bar{x} - \bar{y}||\bar{z} - \bar{p}| &> d(x, y)d(z, p) \\ &= d(x, z)d(p, y) + d(x, p)d(y, z) \\ &= |\bar{x} - \bar{z}||\bar{p} - \bar{y}| + |\bar{x} - \bar{p}||\bar{y} - \bar{z}| \end{aligned}$$

which contradicts the Ptolemaic inequality in \mathbb{R}^2 . Thus, we can apply the rigidity part of the subembedding theorem and obtain in particular that the triangle Δ is flat. This finishes the proof. \square

Remark 6.3. We claim that if x, y are points in $M \setminus \{p\}$ and there is an i_p -geodesic in $\text{Inv}_p(M)$ from x to y , then x, y , and p necessarily lie on a common geodesic segment.

We prove (the contrapositive of) our claim. Suppose x, y, p do not lie on a common geodesic. As in the proof of Proposition 6.2, consider the nondegenerate geodesic triangle $\Delta = T(x, y, p)$. We can again pick a point z such that (5) holds.

Now, examining the proof of Proposition 3.1, we see that in the intersecting case, we can only have (5) if the corresponding equation also holds in the Euclidean comparison picture. By Ptolemy's theorem, this equation can only hold when the four points \bar{x}, \bar{y} ,

\bar{z} , and \bar{p} all lie on a common line or circle. The four points cannot lie on a common line because $T(x, y, p)$, and so also $T(\bar{x}, \bar{y}, \bar{p})$, is nondegenerate. Therefore we can assume that they lie on a common circle. This rules out the possibility that \bar{z} , \bar{p} , and either \bar{x} or \bar{y} lie on a common line. Thus the point \bar{u} (mentioned in the proof for the intersecting case) cannot be \bar{x} or \bar{y} , and so $u \neq x, y$. We can therefore apply Lemma 4.3 to deduce that $d(u, p) < |\bar{u} - \bar{p}|$. This strict inequality leads to a strict version of (3), and so we cannot have equality in Ptolemy's inequality.

In the nonintersecting case, we got the stronger inequality

$$d(x, y)(d(z, x) + d(x, p)) \leq d(x, z)d(y, p) + d(x, p)d(y, z).$$

When we swivel to reduce to Case 1, as described in the proof of Proposition 3.1, the right-hand side gets strictly smaller and the left-hand side remains unchanged, so (5) must fail in this case also.

In view of the above claim, the question naturally arises as to what happens when x, y, p lie on a common geodesic. Suppose that $x \in [p, y]$, $x \neq p, y$. According to (4.2) in [8], the length $\ell_p(\gamma)$ of a path γ with respect to i_p is given by

$$\ell_p(\gamma) = \int_{\gamma} \frac{|dx|}{d(x, p)^2},$$

where $|dx|$ is the length element for (X, d) . Thus

$$\ell_p([x, y]) = \int_{d(x, p)}^{d(y, p)} \frac{dt}{t^2} = \frac{1}{d(x, p)} - \frac{1}{d(y, p)} = \frac{d(y, p) - d(x, p)}{d(x, p)d(x, p)} = i_p(x, y).$$

Using a similar analysis to the proof of Theorem 6.1, the reader can check that $\ell_p(x, y) > i_p(x, y)$ if $p \in [x, y]$, $p \neq x, y$.

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