SEQUENCES OF REVERSE MINKOWSKI TYPE

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Abstract

We study sequences of positive numbers satisfying a reverse Minkowski condition. In particular, we classify those monotonic decreasing sequences which can be rearranged to satisfy such a condition.

1. Introduction

In contrast to infinite sequences, the l^p norms of a finite sequence are all comparable. For infinite sequences, one can only say that the l^p norm is a decreasing function of $p > 0$. In fact, the assumption that two distinct l^p norms of arbitrary contiguous blocks of terms of a fixed sequence are uniformly comparable seems to be a rather strong constraint on the sequence. Non-negative sequences whose non-increasing rearrangements decrease at a geometric or faster rate are easily seen to be of this type (see Lemma 2.1), but there are others, as we shall see, whose non-increasing rearrangements decrease much more slowly; in fact, it follows from Corollary 2.7 that $\sum_{n=1}^{\infty} n^{-q}$ can be rearranged to form such a sequence whenever $q > 1/p$. Producing such examples is a non-trivial exercise since, intuitively, they have to mix the 'large' and 'small' terms in a rather intricate way. In this paper, we shall give simple criteria (see Theorem 2.6) by which one can decide whether or not a given decreasing sequence can be rearranged to produce such a sequence, together with an algorithm for constructing such a rearrangement when it is possible.

From now on, sequences are always assumed to be non-negative with at least one non-zero term. We denote sequences by capital letters and their terms by the corresponding lower-case letters (for example, $A = (a_k)$). Binary operations and relations applied to sequences are to be interpreted in a pointwise sense. For instance, $A \geq 0$ means $a_k \geq 0$ for all k, and A^p is the sequence (a_k^p). If R is a rearrangement of the positive integers, A_R denotes the induced rearrangement of A whose kth term is a_{r_k}. If $1 \leq n < \infty$, $n \leq m \leq \infty$, we call the (possibly finite) sequence $(a_k)_{k=n}^{m}$ a *block*

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of A, and we define $\|A\|_{l^p(n,m)} = \left(\sum_{k=n}^m |a_k|^p\right)^{1/p}$ if $0 < p < \infty$ and $\|A\|_{l^\infty(n,m)} = \max_{n\le k\le m} |a_k|$. As usual, we write $\|A\|_{l^p} = \|A\|_{l^p(1,\infty)}$. Given a bounded sequence A, we denote by $A^* = (a_n^*)$ the non-increasing rearrangement of A.

Suppose $1 < r \le \infty$. It follows from a rather general version of Minkowski's inequality that $\|B\|_{l^r(n,m)} \le \|B\|_{l^1(n,m)}$ (this elementary fact can also be shown in other ways, but the reason given here justifies coining the term "reverse Minkowski condition" below). If $0 < p < q \le \infty$, it follows that $\|A\|_{l^q(n,m)} \le \|A\|_{l^p(n,m)}$, simply by letting $B = A^p$. We define $RM_{p,q}$ to be the class of bounded sequences A that, for some constant $C > 0$, satisfy the reverse Minkowski condition

$$\|A\|_{l^p(n,m)} \le C\|A\|_{l^q(n,m)}, \qquad \text{for all } 1 \le n \le m < \infty. \tag{1.1}$$

We define $RRM_{p,q}$ to be the class of rearrangements of $RM_{p,q}$ sequences. The reason we restrict our study to bounded sequences is that unbounded sequences always have rearrangements satisfying (1.1) for any fixed $0 < p < q \le \infty$, as the reader can readily verify.

The reverse Minkowski condition bears a certain resemblance to the well-known reverse Hölder condition (initially investigated in [3] and [1] ; see [2] for a more recent account). Roughly speaking, the reverse Hölder condition, which is most often defined for a weight w on Euclidean space, says that the fraction of a cube Q where a weight is much bigger than its average on Q must be quite small. By contrast, the reverse Minkowski condition roughly says that the number of terms which are comparable to the largest term in a block must be small in number. Since this is a lower bound on variability rather than an upper bound, one would expect objects satisfying such a condition to be harder to describe. Thus, although one could define an analogous condition on Euclidean space, we shall only attempt in this paper to investigate it in the simpler setting of sequences.

(1.1) implies the limiting inequalities $\|A\|_{l^p(n,\infty)} \le C\|A\|_{l^q(n,\infty)}$, but the converse is false. For example, if we modify the sequence (2^{-k}) by inserting k new terms, all equal to 2^{-2k}, between each pair of old terms 2^{-k} and $2^{-(k+1)}$, the resulting sequence is not in $RM_{p,q}$ for any $p > 0$, but it satisfies all reverse Minkowski inequalities over infinite blocks.

Suppose $0 < p < q \le \infty$. If $A \in RM_{p,q}$, then

$$\|A\|_{l^p(n,m)} \le C\|A\|_{l^q(n,m)} \le C\|A\|_{l^\infty(n,m)}^{1-p/q}\|A\|_{l^p(n,m)}^{p/q}$$

and so $\|A\|_{l^p(n,m)} \le C^{q/(q-p)}\|A\|_{l^\infty(n,m)}$. Thus $RM_{p,q} = RM_{p,\infty}$, so we shall drop the q subscript in $RM_{p,q}$ and $RRM_{p,q}$ from now on. It follows that $RM_p \subset l^p$. Also, it is clear that $RM_q \subseteq RM_p$ if $0 < q < p$, and that $A \in RM_p$ if and only if $A^p \in RM_1$. Thus, to understand RM_p and RRM_p, it suffices to study RM_1 and RRM_1.

2. Main results

If $A \in RM_1$, we denote by C_A the smallest constant C for which $\|A\|_{l^1(n,m)} \le C\|A\|_{l^\infty(n,m)}$ (for all $0 < n < m$). Clearly $C_A \ge 1$, with equality if and only if A has only one non-zero term. We now find a simple necessary and sufficient condition for a monotonic decreasing sequence to be in RM_p.

Lemma 2.1. *Suppose $A = A^*$ and $0 < p < \infty$. Then $A \in RM_p$ if and only if there exist constants $C > 0$ and $0 < t < 1$ such that*

$$a_m \le Ct^{m-n}a_n \qquad \text{for all } 0 < n < m. \tag{2.2}$$

Furthermore, if $A \in RM_p$, then all rearrangements of A are in RM_q for all $q > 0$.

Proof. Without loss of generality we may assume $p = 1$. If $A = A^* \in RM_1$, then $s_n \equiv \sum_{k=n}^{\infty} a_k \le C_A a_n$ for all n. If we write $t = 1 - C_A^{-1}$, then $s_{n+1} \le ts_n$, and so

$$a_m \le s_m \le t^{m-n}s_n \le C_A t^{m-n}a_n.$$

The converse is even easier, as we do not need monotonicity. If $a_m \le Ct^{m-n}a_n$ then, for any $0 < q < \infty$,

$$\sum_{k=n}^{m} a_k^q \le C^q a_n^q \sum_{k=0}^{m-n} t^{kq} \le \frac{C^q a_n^q}{1-t^q}.$$

For the second statement, it suffices to show that if $A \in RM_1$ is monotonic decreasing, and $B = A_R$ is a rearrangement of A, then $B \in RM_1$ also. Among the set of integers i for which $\|B\|_{l^\infty(n,m)} = b_i = a_{r_i}$, let j be the one that minimises r_i. Then

$$\sum_{k=n}^{m} b_k \le \sum_{k=r_j}^{\infty} a_k \le C_A a_{r_j} = C_A b_j. \quad \square$$

The situation for non-monotonic sequences is quite different. The monotonic rearrangement of a general RM_p sequence does not have to satisfy (2.2), although it does satisfy a weaker size condition (see Proposition 2.4). Also, an RM_p sequence is not necessarily in RM_q for any fixed $q < p$ (see Corollary 2.9). Finally, the invariance of the RM_p condition under rearrangements completely breaks down for non-monotonic sequences, as the following example shows.

Example 2.3. Let $B = 2^{-A} = (2^{-a_k})$, where

$$a_k = \begin{cases} 2^i, & \text{if } 2^i - i < k \le 2^i,\ i \ge 3 \\ k, & \text{otherwise.} \end{cases}$$

Let us call a_k a *deviant term* if $a_k \neq k$. Then B is a monotonically decreasing sequence, but $B \notin RM_p$ for any $p > 0$, because of the long blocks of equal terms. However, there exists a rearrangement A' of A such that $B' = 2^{-A'} \in RM_p$ for all $p > 0$. Specifically, we claim that A' can be chosen to be any rearrangement of A in which each deviant term is surrounded by two smaller non-deviant terms. For example, doing this with 'minimal rearranging' yields

$$A' = (1, 2, 3, 8, 4, 8, 5, 8, 9, 16, 10, 16, 11, 16, 12, 16, 17, \ldots, 23, 32, 24, 32, \ldots).$$

Note that in A', the first instance of 2^k occurs after the term $2^k - 2k + 1$, for all $k > 2$.

To prove the claim, note first that $B \leq S$, where $S = (2^{-k})$. Also $b_k = s_k$ unless $-\log_2 b_k$ is a deviant term. By Lemma 2.1, $S_R \in RM_p$ for any rearrangement R of the positive integers. Choose R so that $B_R = B'$ is a rearrangement of the above type. Now, $\|B_R\|_{l^p(n,m)} \leq \|S_R\|_{l^p(n,m)} \leq C\|S_R\|_{l^\infty(n,m)}$, and $\|S_R\|_{l^\infty(n,m)} = \|B_R\|_{l^\infty(n,m)}$ except if $n = m$ and a_{m_r} is a deviant term. But this exceptional case is trivial and so we are done.

Proposition 2.4. *Suppose $A \in RM_1$. If $t = 1 - C_A^{-1}$, and $d_j = \sum_{i=2^j}^{2^{j+1}-1} a_i^*$, then $d_m \leq C_A t^{m-n} d_n$.*

Proof. Let $s_j \equiv \sum_{i=j}^\infty d_i$. We claim that $s_j \leq C_A d_j$, for all $j \geq 0$. Clearly, $s_0 = \|A\|_{l^1} \leq C_A \|A\|_{l^\infty} = C_A d_0$. Suppose $j > 0$. Then s_j is a sum of terms over the 2^j blocks of A (some of which may be empty) obtained by removing the $2^j - 1$ largest terms from the sequence A. Let d_j' be the sum of the largest terms in each of these blocks. Adding the corresponding sides of the reverse Minkowski inequalities over each of these blocks, we get $s_j \leq C_A d_j' \leq C_A d_j$, as required.

It now follows that $s_{j+1} = s_j - d_j \leq (1 - C_A^{-1}) s_j = t s_j$ for all $j > 0$ and so, for all $0 < n < m$,

$$d_m \leq s_m \leq t s_{m-1} \leq \cdots \leq t^{m-n} s_n \leq C_A t^{m-n} d_n. \quad \square$$

Corollary 2.5. *If $A \in RM_p$ for some $p > 0$, then $A \in RM_q$ for some $q < p$.*

Proof. Without loss of generality, we may assume $p = 1$. Suppose n, m are fixed but arbitrary. Let $A_{n,m}$ be the sequence whose kth term is a_k if $n \leq k \leq m$, and 0 otherwise. Also let (b_k) be the non-increasing rearrangement of $A_{n,m}$, and let $d_j = \sum_{k=2^j}^{2^{j+1}-1} b_k$. Clearly $C_{A_{n,m}} \leq C_A$ and so Proposition 2.4 tells us that $d_j \leq C_A t^j d_0$, where $t = 1 - C_A^{-1}$.

By an easy calculus argument, we see that if $x, y \geq 0$, $q < 1$, then $x^q + y^q \leq 2^{1-q}(x+y)^q$. Iterating this inequality, we get that $\sum_{k=2^j}^{2^{j+1}-1} b_j^q \leq 2^{j(1-q)} d_j^q$. Let us choose $q < 1$, but so close to 1 that $r \equiv 2^{1-q} t^q < 1$. Then

$$\|A\|_{l^q(n,m)}^q = \sum_{k=0}^{\infty} b_k^q \leq \sum_{j=0}^{\infty} 2^{j(1-q)} d_j^q \leq C_A^q d_0^q \sum_{j=0}^{\infty} r^j = \frac{C_A^q d_0^q}{1-r} = \frac{C_A^q \|A\|_{l^\infty(n,m)}^q}{1-r},$$

as required. $\square$

The converse of Proposition 2.4 is false. The dyadic sums of a non-negative bounded sequence or of its monotonic decreasing rearrangement cannot alone determine whether or not the sequence is in RM_1 (for example, the monotonic sequence B of Example 2.3 is not in RM_p for any $p > 0$, while its minimal rearrangement B' has the same dyadic sums and is in RM_p for all $p > 0$). However, the dyadic sums of a monotonic sequence are sufficient to decide if the sequence can be rearranged into a RM_1 sequence, as the following theorem reveals.

Theorem 2.6. *Suppose $A = A^*$ and let $d_j \equiv \sum_{k=2^j}^{2^{j+1}-1} a_k$. Then the following are equivalent.*

(i) *$A \in RRM_1$.*
(ii) *There exists $C < \infty$ and $0 < t < 1$ such that $d_m \leq C t^{m-n} d_n$, for all $m > n > 0$.*
(iii) *There exists $C < \infty$ and $s > 1$ such that $a_m \leq C(n/m)^s a_n$, for all $m > n > 0$.*

Note that conditions (ii) and (iii) limit how many similar-sized terms can be in initial segments of A. Consider, for instance, monotonic sequences A such that $a_k = 2^{-b_j}$ for all $b_j < k \leq b_{j+1}$, where B is some increasing sequence. If $b_j = c^j$ for any $c > 1$, then A^p satisfies (iii) for all $p > 0$, while if $b_j = j!$, then A^p violates (iii) for all $p > 0$. Note also that $(n^{-s}) \in RRM_1$ whenever $s > 1$. As an immediate corollary, we have the following characterisation of RRM_p sequences.

Corollary 2.7. *Suppose $0 < p < \infty$ and that A is a bounded sequence. Then $A \in RRM_p$ if and only if there exist $C < \infty$, $s > 1/p$ such that $a_m^* \leq C(n/m)^s a_n^*$ for all $m > n > 0$, where A^* is the decreasing rearrangement of A.*

Proof of Theorem 2.6. (i) implies (ii) by Proposition 2.4. Let us now assume (ii) and prove (iii). Define integers i and j by the inequalities $2^{i-1} < n \leq 2^i$ and $2^j - 1 \leq m < 2^{j+1} - 1$. Letting $s = 1 - \log_2 t > 1$, (ii) and monotonicity imply that

$$a_m \leq a_{2^j-1} \leq d_{j-1}/2^{j-1} \leq C t^{j-i-1} d_i / 2^{j-1} \leq C \left(\frac{t}{2}\right)^{j-i-1} a_{2^i} \leq C'(n/m)^s a_n.$$

Finally, we assume (iii) and prove (i). We do so first for the case $s > 2$, by considering an ordering of the integers induced by a certain infinite tree structure. Levels 0 through 4 of this tree T are shown in Figure 1. As indicated there, T has 2^j nodes at level j, labelled from right to left by the integers 2^j through $2^{j+1}-1$. Node 1, the single node at level 0, is connected to two nodes at level 1 which we refer to as its "far-right" (node 2) and "far-left" (node 3) offspring. If $k > 0$, node k has either four offspring (if $2^j \le k < 2^j + 2^{j-1}$ for some $j > 0$) or no offspring (if $2^j + 2^{j-1} \le k < 2^{j+1}$ for some $j > 0$). In the former case, its offspring have four consecutive integers as labels. By increasing order of labels, we refer to these nodes as the far-right, near-right, near-left, and far-left offspring of node k, for obvious diagrammatical reasons.

For any $k > 0$, we define $S_L(k)$, the set of 'far-left descendents' of k, to be empty if k does not have a far-left daughter, and otherwise to consist of k's far-left daughter and all descendents of that daughter. Similarly, we define $S_l(k)$, $S_r(k)$, and $S_R(k)$ to be the set of near-left, near-right, and far-right descendents of k. We also define $S(k) = S_L(k) \cup S_l(k) \cup S_r(k) \cup S_R(k) \cup \{k\}$.

Any ordering $\prec$ on $\mathbf{N}$ can be extended to a partial ordering (which we also denote by $\prec$) on subsets of $\mathbf{N}$ simply by writing $U \prec V$ if U and V are two sets of positive integers such that $u \prec v$ for every $u \in U$, $v \in V$. Using this notation, we now define $\prec$ to be the unique ordering on $\mathbf{N}$ whose extension satisfies

$$S_L(k) \prec S_l(k) \prec \{k\} \prec S_r(k) \prec S_R(k),$$

for all $k \in \mathbf{N}$. If we now write the positive integers in $\prec$-ascending order, we get the following rearrangement of $\mathbf{N}$:

$$R = (3, 1, 7, 6, 2, 15, 14, 5, 13, 12, 31, 30, 11, 29, 28, 27, 26, 10, 25, 24, 4, 63, \ldots).$$

To see that $\prec$ induces a rearrangement of $\mathbf{N}$, we need only show that $S_L(k)$ and $S_l(k)$ are always finite sets. By construction, the number of nodes at level j with descendents at level $i > j$ halves each time i is incremented by one, until finally there is only one such node (namely $k = 2^j$) for all $i \ge 2j$. The finiteness of $S_L(k)$ and $S_l(k)$ follows immediately since these sets cannot contain powers of 2.

We now show that $B = A_R \in RM_1$. Let $L_j = \{k \in \mathbf{N} \mid 2^j \le k < 2^{j+1}\}$, the set of all nodes at level j. Suppose $k \in L_{j_0}$ and $j > j_0$. Clearly $S(k) \cap L_j$ has at most 4^{j-j_0} nodes. Also, it is inductively clear that if $k \ne m \in S(k)$, then $m \ge 2k$. It follows that if $m \in S(k) \cap L_j$, then $m \ge 2^{j-j_0} k$. Since $s > 2$, (iii) implies that

$$\sum_{i \in S(k)} a_i = \sum_{j=j_0}^{\infty} \sum_{i \in S(k) \cap L_j} a_i \le C a_k \sum_{j=j_0}^{\infty} 4^{j-j_0} / 2^{(j-j_0)s} \le C' a_k. \tag{2.8}$$

We now fix a non-empty block $F = (r_i)_{i=n}^{m}$ of R. Suppose $j = j_0$ is the smallest integer for which $F \cap L_j$ is non-empty. The set $F_0 = F \cap L_{j_0}$ is a block of at most four consecutive integers. We would like to be able to say that if $i \in F$ then $i \in S(k)$ for some $k \in F_0$, but this is not necessarily true. If $k_1 \equiv \min_{k \in F_0} k > 2^{j_0}$, then i may lie in $S_R(k_1 - 1) \cup S_r(k_1 - 1)$ and, if $k_2 = \max_{k \in F_0} k < 2^{j_0+1} - 1$, i may be in $S_L(k_2 + 1) \cup S_l(k_2 + 1)$. We therefore define F_0' to be the smallest superset of F_0 which also contains any right offspring of $k_1 - 1$ and any left offspring of $k_2 + 1$ that belong to F (e.g. if $F_0 = \{9\}$, then $F_0' \subseteq \{9, 18, 19, 24, 25\}$). Now, any $i \in F$ lies in $S(k)$, for some $k \in F_0'$. Using (2.8), we deduce that $\sum_{i=n}^m b_i \le C' \sum_{i \in F_0'} a_i$. Clearly F_0' has at most eight elements (in fact, a little further reflection reveals that it has at most six), so the required reverse Minkowski inequality follows.

If $1 < s \le 2$, the argument breaks down because the geometric sum in (2.8) is no longer convergent. If we replace our previous tree with a standard binary tree, the analogous version of (2.8) is valid for any $s > 1$, since we can replace the 4^{j-j_0} factor in the geometric sum by 2^{j-j_0}. Unfortunately, each node in a binary tree has infinitely many descendents on both the left and the right, so such a tree does not induce a (sequential) rearrangement of $\mathbf{N}$. We instead use a hybrid tree in which, at each level, either every node has exactly two offspring ('far left' and 'far right') or, as with our original example, the leftmost half of the nodes are childless and the rightmost half of the nodes have four offspring each. We refer to the former type of level as a *binary level* and the latter type as a *non-binary level*. Note that, with one node at level 0, such trees have 2^j nodes at level j for all $j \ge 0$, and they are uniquely determined once we specify which levels are binary.

For fixed $m > 1$, we denote by T_m the tree which has one non-binary level below each $m-1$ binary levels (so the jth level is non-binary if and only if $j+1$ is divisible by m). As before, we label the nodes in the jth level with consecutive integers between 2^j and $2^{j+1} - 1$ as we traverse it from right to left, and hence get an order on the integers. The non-binary levels ensure that the sets of left descendents, $S_L(k)$ and $S_l(k)$, are finite for all k. It follows that the T_m-induced order of $\mathbf{N}$ actually gives a rearrangement R_m of $\mathbf{N}$. It is not hard to see that if $k \in L_{j_0}$ then $S(k) \cap L_j$ has less than $2^{(j-j_0)(1+1/m)+1}$ terms and that $m \ge 2^{j-j_0}k$ if $m \in S(k) \cap L_j$. Thus $\sum_{i \in S(k)} b_i \le C b_k$, as long as $s > 1 + 1/m$. By choosing large enough m, we see that (iii) implies (i) for any $s > 1$ (alternatively, a single tree will work for all s, if the number of binary levels between successive non-binary levels tends to infinity as one progresses down the tree). $\square$

The following corollary (which should be contrasted with 2.5) is now immediate, since $B = (n^{-1/q}) \in RRM_p \setminus RRM_q$, for any $0 < q < p$.

Corollary 2.9. *Given $0 < q < p$, there exists a bounded sequence A in $RM_p \setminus RM_q$.*

As Lemma 2.1 and Example 2.3 indicate, there are many ways one can rearrange a sequence which decreases quickly most of the time, in order to produce a sequence in RM_p for all $p > 0$. We shall now show that there is much less freedom when dealing with sequences whose monotonic rearrangements decrease slowly—in fact any such RM_1 sequence is naturally associated with one of a rather general class of trees that include all the trees T, T_m used in the proof of Theorem 2.6.

Let us begin with a couple of definitions. By a *rearrangement tree* we shall mean a tree with the following properties.

(a) It has 2^j nodes at level j, $j \geq 0$, each of which have a unique label which is an element of L_j (we do not insist on any specific numeric ordering of the labels). We identify the node with its label.
(b) Each node has a unique mother-node at some earlier level (not necessarily one level removed) and the offspring of a node k are arranged from left to right (a certain number on the left of k, the remainder on the right).
(c) Each node has a finite number of left descendants.

As the name suggests, any rearrangement tree induces a rearrangement of the positive integers by iterating the basic rule that the offspring of a node k are ordered from left to right, and that k falls between its left and right offspring. This rule allows one to find the relative order of the integers less than 2^{j+1} by considering only levels 0 through j. As one considers more levels, the larger integers are inserted in this list; we get a rearrangement rather than a more general reordering because (c) guarantees that any initial segment of the list receives only a finite number of later insertions. For example the tree in Figure 2 induces the following relative order on the first 31 positive integers:

$$\begin{aligned}&2 \prec 1 \prec 6 \prec 11 \prec 20 \prec 5 \prec 30 \prec 3 \prec 8 \prec 21 \prec 19 \prec 12 \prec 17 \prec 7 \prec 16 \prec 28 \prec \\ &\prec 9 \prec 22 \prec 29 \prec 13 \prec 23 \prec 27 \prec 15 \prec 18 \prec 4 \prec 24 \prec 10 \prec 25 \prec 31 \prec 14 \prec 26\end{aligned}$$

We say that a rearrangement tree is an *M-tree* if each node has at most M offspring at any one subsequent level (although the total number of its offspring may be infinite).

We claim that if A is any decreasing sequence satisfying condition (iii) of Theorem 2.6 for sufficiently large $s = s(M)$, then $A_R \in RRM_1$, where R is the rearrangement of the integers induced by an M-tree. First note that the within-level right-to-left labelling of the nodes at level j is employed in Theorem 2.6 only to minimise C' in (2.8); the more general labelling schemes we are now allowing simply require C' to be multiplied by a factor 2^s (since $m \in S(k) \cap L_j$ now only implies that $m \geq 2^{j-j_0-1}k$). In fact an examination of the proof of Theorem 2.6 reveals that the only characteristics needed of our tree to make the argument valid (for sufficiently large m) are that there exist numbers $K_0, K_1, K_2 > 0$ such that the number of descendents of a level-j_0 node at level j is at most $K_0 K_1^{j-j_0}$, and that in the induced rearrangement of $\mathbf{N}$, there are no more than K_2 nodes from level j in any block that has no nodes at level k for all $k < j$; this is clearly the case for M-trees (with $K_0 = K_1 = M, K_2 = 2M$).

Conversely suppose that $A \in RM_1$ and that there exists $c > 0$, $s < \infty$, such that for all $m > n > 0$, $a_m^* \geq c(n/m)^s a_n^*$. In particular, pairs of terms from a single dyadic block of A^* have bounded quotients. Let R be a rearrangement of $\mathbf{N}$ for which $A = A^*_R$. Since $A \in RM_1$, any block of A containing more than a fixed number of terms with index in L_j (all of which are approximately equal), must also contain a term with index in L_k for some $k < j$. But it is easy to show that any such rearrangement R must be induced by an M-tree: we build our tree one level at a time, letting the new entries between the two nodes $a \prec b$ be left daughters of b or right daughters of a (it does not matter which we choose to do as long as we choose an order consistent with the relative order of the newly inserted nodes). Thus we have shown:

Theorem 2.10. *If R is the rearrangement induced by an M-tree, then there exists $s_0 = s_0(M) > 0$ such that $A_R \in RM_1$ whenever $A = A^*$ satisfies $a_m \le C(n/m)^s a_n$, for all $m > n > 0$, and some $s > s_0$, $C > 0$. Conversely, if $A \in RM_1$ and if there exists an $s > 1$ such that $a^*_m \ge c(n/m)^s a^*_n$ for all $m > n > 0$, then $A = A^*_R$ for some rearrangement induced by an M-tree, $M = M(c, s)$.*

We now consider how conditions (i)–(iii) in Theorem 2.6 are related when we drop the assumption that A is monotonic decreasing. It is obvious that (iii) implies (ii), and it also implies (i), as we did not use monotonicity when proving this implication in the theorem. However, (i) does not imply (ii) (let alone (iii)), as it is easy to rearrange any monotonic sequence in RRM_1 to get a sequence whose dyadic sums do not satisfy (ii). One might hope that (ii), or perhaps some faster rate of decay of the dyadic sums, implies (i). Perhaps surprisingly, the answer is always negative, as the following proposition indicates.

Proposition 2.11. *Given any sequence $T > 0$ there is a sequence $A > 0$ whose dyadic sums $d_j \equiv \sum_{k=2^j}^{2^{j+1}-1} a_k$ satisfy $d_{j+1}/d_j \le t_j$ but which is not in RRM_1.*

Proof. To prove the result for arbitrary $T > 0$, it suffices to prove it for some $T' \le T$. We may therefore assume without loss of generality that $t_i < 1$ for all i and that for $n \ge 3$,

$$t_{2^n-n} < 2^{n-2^n}, \tag{2.12}$$

$$t_{2^n-n} < \prod_{i=2^n-n+1}^{2^n-1} \frac{t_j}{2}. \tag{2.13}$$

Writing $u_j = \prod_{i=1}^{j-1}(t_i/2)^2$, we define B to be the monotonic sequence whose terms are constant on dyadic blocks, and whose dyadic sums $e_j \equiv \sum_{i=2^j}^{2^{j+1}-1} b_i = 2^j b_{2^j}$ are given by the formula

$$e_j = \begin{cases} u_{2^n}, & \text{if } 2^n - n < j \le 2^n, \text{ for some } n > 2, \\ u_j, & \text{otherwise.} \end{cases}$$

For the rest of the proof, n is any integer greater than 2. We define the nth *plateau*, $P(n) = \{j \in \mathbf{N} \mid 2^n - n < j \leq 2^n\}$, and the nth *pre-plateau*, $PP(n) = \{j - n \mid j \in P(n)\}$. We also write $P(*) = \bigcup_{n>2} P(n)$ and $PP(*) = \bigcup_{n>2} PP(n)$. Given any sequence, we call one of its terms an nth *plateau term* if the term's index is in $L_j \equiv \{k \in \mathbf{N} \mid 2^j \leq k < 2^{j+1} - 1\}$ for some $j \in P(n)$; we define nth *pre-plateau terms* similarly. Away from the plateaus, the dyadic sums e_j decrease at a faster rate than required, but $e_{j+1} = e_j$ whenever $j, j+1 \in P(*)$.

There is a much larger than necessary decrease for $j = 2^n - n$, which we shall exploit. To do so, we first define a perturbation B' of B by changing a single term b_{k_j} in the jth dyadic block (we may choose $k_j = 2^j$) whenever $j \in PP(*)$. Specifically we choose

$$b'_{k_j} \equiv c_j = u_{2^n - n} \prod_{i=2^n-n}^{j+n-1} \frac{t_j}{2}.$$

Note that we have the following important facts: if $j \in PP(n)$, then (2.12) implies that c_j is less than any nth pre-plateau term of B, while (2.13) implies that c_j is larger than any nth plateau term.

For all $j \in PP(n)$, $n > 2$, we swap b'_{k_j} with some term in the $(j+n)$th dyadic block of B', and refer to the resulting rearrangement of B' as A. Let us denote by d_j the jth dyadic sum of A, and write $r_j = d_{j+1}/d_j$. We claim that $r_j < t_j$ for all j. If neither j nor $j+1$ are in $P(*) \cup PP(*)$, then $r_j = t_j^2/4 < t_j$; similarly, if $j = 2^n$, then $r_j < t_j^2/4 < t_j$. If $j = 2^n - n$, then

$$r_j < \frac{c_{j-n+1} + u_{2^n}}{d_j} < \frac{2t_j + t_j^2}{4(1 - 2^{-j})} < t_j.$$

Similarly, if $j+1 \in PP(n)$, then $r_j < (t_j^2/4)(1-2^{-j})^{-1} < t_j$. Finally, if $j, j+1 \in P(n)$, then $r_j < (c_{j+1} + u_{2^n})/c_j < 2t_j/2$.

By construction, the altered terms fit in the last n spots of the $(2^n - n)$th dyadic block of A^*, and the plateau terms of A^* are exactly the same as those of B' (or B). Thus A^* has n consecutive equal dyadic sums, for arbitrarily large n. By Theorem 2.6, $A \notin RRM_1$. $\square$

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