Distributive algebras, isoclinism, and invariant probabilities

STEPHEN M. BUCKLEY

ABSTRACT. We develop a basic theory of distributive algebras, a certain class of universal algebras that generalize the class of (associative and nonassociative) rings. We then define and investigate isoclinism for distributive algebras—this is an equivalence relation among distributive algebras of a particular type—and we relate isoclinism to ring theory via isologism with respect to varieties of (possibly nonassociative) rings. Associated with any given ring variety is a map from rings to distributive algebras of a particular type, and we say that rings are isologic with respect to this variety if the associated distributive algebras are isoclinic. Certain probability functions on finite distributive algebras are invariant under isoclinism. These invariants allow us to derive some combinatorial consequences in ring theory by using an appropriate isologism.

1. INTRODUCTION

Isoclinism for groups is an equivalence relation that was introduced by Hall [10] and is used widely in the literature of group theory. The more general concept of group isologism, which is essentially isoclinism with respect to a variety of groups was also introduced by Hall [11]. For more on group isoclinism and isologism, see for instance [1] and [12], respectively.

There are existing notions of isoclinism for rings and Lie algebras due to Kruse and Price [14] and Moneyhun [15], respectively. In [3, Section 3], a new type of isoclinism was introduced. The major difference between this concept and the earlier ones is that additive group isomorphisms rather than ring homomorphisms were employed in the definition. The extra flexibility provided by this difference was an essential ingredient for the investigation of the commuting probability of a finite ring in that paper.

In the current paper, we develop a much more general theory of isoclinism, again using group isomorphisms. We relate isoclinism to ring theory via isologism with respect to varieties of (possibly nonassociative) rings. Any given isologism sets up a map from rings to distributive algebras. We then use our theory to obtain further results concerning the commuting probability and related concepts of combinatorial ring theory.

The setting for our theory of isoclinism is a class of universal algebras that we call *distributive algebras*. Such algebras include as special cases possibly nonassociative rings (abbreviated in the rest of the paper as *PN rings*), as well as Lie and Jordan triple systems and Jordan $*$-triple systems. We develop a rudimentary theory of such algebras sufficient for the needs of our subsequent theory of isoclinism.

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Rather than saying any more about isoclinism and isologism at this point, we will instead state some ring theoretic consequences. First we need to define some probability functions on the class of finite PN rings, and associated *spectra*.

We use a formal "noncommutative polynomial" $f(X,Y) = aXY + bYX$, where $a,b \in \mathbb{Z}$, as a symbol of a function $f^R : R\times R \to R$, $f^R(x,y) := axy+byx$, defined on an arbitrary PN ring R. For such an f, and a PN ring R of finite cardinality, we define

$$\mathrm{Pr}_f(R) := \frac{|\{(x,y) \in R\times R :\ f^R(x,y) = 0\}|}{|R|^2}, \tag{1.1}$$

where $|S|$ denotes the cardinality of a set S. Whenever $\mathcal{C}$ is a class of finite PN rings, we define the associated *f-spectrum* $\mathfrak{S}_f(\mathcal{C}) \subseteq \mathbb{Q}\cap(0,1]$ by

$$\mathfrak{S}_f(\mathcal{C}) := \{\mathrm{Pr}_f(R) \mid R \in \mathcal{C}\}\,.$$

Our first result says that associativity makes no difference for any of these spectra; in fact we will see in Theorem 5.9 and Remark 5.10 that the same conclusion holds for some other function symbols f. Throughout this paper, we allow rings and PN rings to be non-unital.

Theorem 1.1. *Let $f(X,Y) := aXY + bYX$ for some $a,b\in\mathbb{Z}$, and let $\mathcal{C}$ and $\mathcal{C}_{\mathrm{pn}}$ be the classes of all finite rings, and all finite PN rings, respectively. Then $\mathfrak{S}_f(\mathcal{C}) = \mathfrak{S}_f(\mathcal{C}_{\mathrm{pn}})$.*

We use special names and notation for $\mathrm{Pr}_f(R)$ and $\mathfrak{S}_f(\mathcal{C})$ in connection with three fundamental functions f of this type. For $f(X,Y) = XY - YX$, we speak of the *commuting probability* $\mathrm{Pr}_{\mathrm{c}}(R)$ and *commuting spectrum* $\mathfrak{S}_{\mathrm{c}}(\mathcal{C})$; for $f(X,Y) = XY + YX$, the *anticommuting probability* $\mathrm{Pr}_{\mathrm{ac}}(R)$ and *anticommuting spectrum* $\mathfrak{S}_{\mathrm{ac}}(\mathcal{C})$; and for $f(X,Y) = XY$, the *annihilating probability* $\mathrm{Pr}_{\mathrm{ann}}(R)$ and *annihilating spectrum* $\mathfrak{S}_{\mathrm{ann}}(\mathcal{C})$.

The next result links spectra for different choices of f.

Theorem 1.2. *Let $f(X,Y) := aXY + bYX$ for some $a,b\in\mathbb{Z}$, and let $\mathcal{C}$ be the class of all finite rings. Then $\mathfrak{S}_{\mathrm{ann}}(\mathcal{C}) \supseteq \mathfrak{S}_f(\mathcal{C})$.*

Our use of isoclinism in this paper establishes the above links between the various spectra *while paying no attention to the precise values that lie in these spectra*. Thus the above results are complementary to those in [3] where we examine only commuting spectra but prove some rather precise results: specifically, we characterize all $t \in [11/32, 1]$ that lie in the commuting spectrum for finite rings, and all $t \in [(p^3+p^2-1)/p^5, 1]$ in the commuting spectrum for rings of order a power of a prime p.

After some preliminaries in Section 2, we discuss distributive algebras in Section 3, and then isoclinism and isologism in Section 4. Finally we discuss isoclinism invariants such as the commuting probability in Section 5, and prove the results stated above.

2. Preliminaries

The unqualified term *ring* always mean an associative ring. Both PN rings and rings are allowed to be nonunital. If p is a prime, a *(PN) $\mathbb{Z}_p$-algebra* is a (PN) ring R such that $pR = \{0\}$, and a *(PN) p-ring* is a (PN) ring whose cardinality is finite and a power of p. More generally a *classical (PN) algebra over a field F* is a vector space over F with a bilinear product (which is associative if the PN qualifier is omitted); again it is not necessarily unital. We use the qualifier *classical* here because we will use the unadorned term *algebra* only in the universal algebra sense of a nonempty set with an associated collection of fundamental operations, as discussed in Section 3.

We write $\mathbb{Z}_n$ for ring of integers mod $n \in \mathbb{N}$, and O_p for the null ring of prime order p.

Let us define a few additive subgroups of $(R,+)$ associated with a PN ring R.

- We inductively define R^n for $n \in \mathbb{N}$: $R^1 = R$ and, for $n > 1$, R^n is the subgroup of R generated by products xy, where either x or y lies in R^{n-1}. Thus for instance the subgroup R^3 is generated by all elements of form $(xy)z$ or $x(yz)$, $x,y,z \in R$, while R^4 is generated by elements of the form $((xy)z)w$, $(x(yz))w$, $x((yz)w)$, and $x(y(zw))$. Note that $u := (xy)(zw) \in R^3$, but it is possible that $u \notin R^4$ if R is nonassociative.
- $[R,R]$, the *commutator subgroup*, is generated (additively) by all *commutators* $[x,y] = xy - yx$.
- $\langle R,R\rangle$, the *anticommutator subgroup*, is generated (additively) by all *anticommutators* $\langle x,y\rangle = xy + yx$.
- $\mathrm{Ann}(R) = \{a \in R \mid ax = xa = 0 \text{ for all } x \in R\}$ is the *annihilator of* R.
- $Z(R) \quad = \{a \in R \mid [a,x] = 0 \text{ for all } x \in R\}$ is the *center of* R.
- $AZ(R) \ = \{a \in R \mid \langle a,x\rangle = 0 \text{ for all } x \in R\}$ is the *anticenter of* R.

Some of the above subgroups behave well under multiplication: in fact, R^n and $\mathrm{Ann}(R)$ are ideals, while $Z(R)$ is a subring. However the other subgroups defined above are not in general closed under multiplication.

3. Distributive algebras

In this section we develop a basic theory of what we call *distributive algebras*: these are universal algebras with certain properties that generalize the class of PN rings. For the general theory of universal algebras, we refer the reader to [8], [9], or [13].

We begin with some preparatory definitions. If S is any set, we define $S^{\times 0} = \{\emptyset\}$, while $S^{\times n}$ is the cartesian product[1] of n copies of S for all $n \in \mathbb{N}$.

An *algebra* A consists of an *underlying set*[2], also denoted A, with an attached set of fundamental operations $g^A : A^{\times n} \to A$; here, the non-negative integer n can depend on g^A and is called the *arity of* g^A. We speak of *nullary*, *unary*, or *binary* operations if $n = 0$, $n = 1$, or $n = 2$, respectively. A nullary operation is a significant constant, such as 0 or 1 in a unital ring.

We often use vector-style notation for the argument list of an operation in an algebra. Thus if g^A is an n-ary operation on A, and if we write $g^A(\underline{x})$ then, unless otherwise stated, $\underline{x} = (x_1,\dots,x_n) \in A^{\times n}$. We call each x_i a *coordinate* of $\underline{x}$ and the *coordinate set of* $\underline{x}$ is $\mathrm{CS}(\underline{x}) = \{x_1,\dots,x_n\}$. The algebras of interest to us are built on an abelian group, and we define sums $\underline{x}+\underline{y}$ by coordinate-wise addition. The case $n = 0$ is special since then $\underline{x} = \emptyset$.

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[1]We write the cartesian product as $S^{\times n}$ rather than S^n because we reserve the latter notation for product ideals in a PN ring, as defined in the previous section.

[2]It is common in the universal algebra literature to distinguish notationally between an algebra and the underlying set. However, since we are mainly interested in algebras whose operations are defined in terms of the operations of an underlying ring, we use the ring theoretic convention of avoiding such notational distinctions. Of course this requires that we distinguish notationally between two distinct algebras that have the same underlying set.

3.1. **Distributive algebras.** Suppose g^A is an n-ary operation on an abelian group $(A,+)$ for some $n \in \mathbb{N}$. We say that g^A is *distributive over addition* if it is multilinear over A as a $\mathbb{Z}$-module. Explicitly, if $\underline{x}, \underline{y}, \underline{z} \in A^{\times n}$ are such that $z_j = x_j + y_j$ for some $1 \le j \le n$, and $z_k = x_k = y_k$ for all $1 \le k \le n$, $k \ne j$, then $g^A(\underline{z}) = g^A(\underline{x}) + g^A(\underline{y})$. This generalizes the usual definition of distributivity of multiplication in a ring.

Definition 3.2. Suppose I is an index set and $\rho : I \to \mathbb{N}$. An (I,ρ)-*algebra* is an abelian group $(A,+)$ with an associated set of $\rho(i)$-ary operations g_i^A on A, $i \in I$, such that g_i^A is distributive over addition whenever $\rho(i) > 0$; (I,ρ) is the *type* of A. A *distributive algebra* means an (I,ρ)-algebra for some type (I,ρ).

If $|I|$ is small, it is convenient to take the index set to be $\{1,\dots,k\}$ and to write the type as $[\rho(1),\dots,\rho(k)]$ instead of (i,ρ). For instance, the concepts of PN rings and [2]-algebras coincide, while a unital PN ring is a $[2,0]$-algebra A such that $g_2^A(\emptyset)$ satisfies the identities $g_1^A(g_2^A(\emptyset),x) = g_1^A(x, g_2^A(\emptyset)) = x$ for all $x \in A$.

If A is an (I,ρ)-algebra and $0 \in \mathrm{CS}(\underline{x})$, $\underline{x} \in A^{\times n}$, then $g_i^A(\underline{x}) = 0$ for all $i \in I$. This follows as for rings by writing $0 = 0 + 0$ and using distributivity.

Sometimes, as in Definition 3.2, we want to discard the nullary operations. We denote by (I_0,ρ_0) the *reduced type corresponding to the type* (I,ρ): this means that the *reduced index set* I_0 consists of all $i \in I$ such that $\rho(i) > 0$, and $\rho_0 := \rho|_{I_0}$. An (I,ρ)-algebra is said to have *reduced type* if $I = I_0$: general rings and PN rings are of reduced type, whereas unital rings and unital PN rings are not if the unity is considered to be part of the structure.

3.3. **Subalgebras and ideals.** We first define subalgebras, homomorphisms, and ideals for distributive algebras. We use the full index set I for defining subalgebras and homomorphisms but only the reduced index set I_0 for defining ideals. This is consistent with the usual convention in the theory of unital rings that subrings contain the same unity as the full ring, and homomorphisms map unity to unity, but of course ideals are not required to contain the unity.

Definition 3.4. An (I,ρ)-algebra B is a *subalgebra* of an (I,ρ)-algebra A, denoted $B \le A$, if $(B,+)$ is a subgroup of $(A,+)$, and each g_i^B is a restriction of g_i^A. The *trivial subalgebra* 0 of A is the one containing only the single element 0.

Definition 3.5. A *homomorphism* (or *isomorphism*) from one (I,ρ)-algebra A to another B is a group homomorphism (or isomorphism) $\phi : (A,+) \to (B,+)$ such that for all $i \in I$ with $n := \rho(i)$, and all $\underline{x} \in A^{\times n}$, $\underline{y} \in B^{\times n}$, with $\phi(x_j) = y_j$ for $j = 1,\dots,n$, we have $\phi(g_i^A(\underline{x})) = g_i^B(\underline{y})$. An *endomorphism* is a homomorphism from an algebra to itself. We define $\mathcal{A}lg_{\mathcal{D}}$ to be the category of all distributive algebras, with homomorphisms as the morphisms.

Definition 3.6. An *ideal in an* (I,ρ)-*algebra* A is a subgroup J of $(A,+)$ with the property that $g_i^A(\underline{x}) \in J$ whenever $i \in I_0$, $\underline{x} \in A^{\times \rho(i)}$, and $\mathrm{CS}(\underline{x}) \cap J$ is nonempty; as always (I_0,ρ_0) is the reduced type corresponding to (I,ρ). We write $J \trianglelefteq A$ or $A \trianglerighteq J$ if J is an ideal in A. An ideal in an (I,ρ)-algebra is implicitly an (I_0,ρ_0)-algebra.

Note that by distributivity, an additive subgroup J generated (additively) by a set S is an ideal if $g_i^A(\underline{x}) \in J$ whenever $i \in I_0$, $\underline{x} \in A^{\times\rho(i)}$, and $\mathrm{CS}(\underline{x}) \cap S$ is nonempty.

The intersection $\bigcap_{j\in J} A_j$ of a collection of additive subgroups A_j of A is itself a subgroup of A (and a subalgebra of A, or an ideal in A, if every A_j is a subalgebra or ideal, respectively). Similarly if each A_j is an additive subgroup (or ideal), and if we define the sum $\sum_{j\in J} A_j$ as the set of finite sums of elements in the individual sets A_j, then we get an additive subgroup (or ideal). When the index set J is empty, an intersection $\bigcap_{j\in J} A_j$ of subgroups means A itself, and a sum $\sum_{j\in J} A_j$ of subgroups means the trivial subgroup.

We now define the two ideals of main interest to us for a given (I,ρ)-algebra A with reduced index set I_0.

Definition 3.7. The *annihilator of* A is $\mathrm{Ann}(A) = \bigcap_{i\in I_0} \mathrm{Ann}(A;i)$, where

$$\mathrm{Ann}(A;i) = \{a \in A \mid \forall\, \underline{x} \in A^{\times\rho(i)} : a \in \mathrm{CS}(\underline{x}) \Rightarrow g_i^A(\underline{x}) = 0\}\,, \quad i \in I_0\,.$$

Definition 3.8. The *product ideal of* A, $\pi(A)$, is the subgroup of $(A,+)$ generated by elements of $\pi(A;i)$, $i \in I_0$, where $\pi(A,i)$ is the subgroup of $(A,+)$ generated by all elements of the form $g_i^A(\underline{x})$, $\underline{x} \in A^{\times\rho(i)}$.

It is readily verified that both $\mathrm{Ann}(A)$ and $\pi(A)$ are ideals in A. It is however easy to construct algebras with more than one operation in which the subgroups $\mathrm{Ann}(A;i)$ and $\pi(A;i)$ fail to be ideals for any particular i. However $\pi(A;i)$ will be of interest later.

A *null algebra* is a distributive algebra A such that $\mathrm{Ann}(A) = A$, or equivalently $\pi(A) = 0$. Null algebras include in particular *null-type algebras*: these are (I,ρ)-algebras where I is the empty set, and so they are simply abelian groups.

We now record a lemma for distributive algebras that generalizes a basic result for rings. The proof is simple and of a standard type, but we include it as an example of the use of these concepts.

Lemma 3.9. *If J is an ideal in an (I,ρ)-algebra A, then the quotient group A/J naturally has the structure of an (I,ρ)-algebra.*

Proof. A/J is an additive group, and we make it into an (I,ρ)-algebra by defining $g_i^{A/J}(\underline{x} + J) = g_i^A(\underline{x}) + J$ for all $\underline{x} \in A^{\times\rho(i)}$, $i \in I$; here $\underline{x} + J$ means $(x_1 + J, \ldots, x_n + J)$. We need to check that this is well-defined for $i \in I_0$. Fixing $i \in I_0$, we let $n := \rho(i)$, $\underline{x} \in A^{\times n}$, and $\underline{y} \in J^{\times n}$. By distributivity, $g_i^A(\underline{x} + \underline{y})$ equals $g_i^A(\underline{x})$ plus a sum of terms of the form $g_i^A(\underline{z})$ where $\mathrm{CS}(\underline{z}) \cap \mathrm{CS}(\underline{y})$ is nonempty. Consequently $g_i^A(\underline{x} + \underline{y}) - g_i^A(\underline{x}) \in J$, as required. Finally, we note that $g_i^{A/J}$ inherits distributivity over addition from g_i^A. $\square$

Whenever we write $g_i^{A/J}$ below, we always mean the natural map as defined in the above proof.

Remark 3.10. Using distributivity as in Lemma 3.9, we see that $g_i^{A/\mathrm{Ann}(A)}$ factors through A to yield a natural map $\tilde{g}_i^A : (A/\mathrm{Ann}(A))^{\times n} \to A$.

Definition 3.11. Suppose A_j is an (I,ρ)-algebra for every j in some nonempty index set J. Let A be the cartesian product of the underlying sets A_j, $j \in J$, and let B be the subset of A consisting of those $(a_j)_{j\in J}$ such that $a_j = 0$ for all except finitely many indices j. The *direct product* $\prod_{j\in J} A_j$ and the *direct sum* $\bigoplus_{j\in J} A_j$ consist of the sets A and B, respectively, with the associated operations g_i^A and g_i^B induced from those of the algebras A_j in a coordinatewise manner. In this way, a direct product of (I,ρ)-algebras is an (I,ρ)-algebra, and a direct sum of a (I,ρ)-algebras is an (I_0,ρ_0)-algebra, where (I_0,ρ_0) is the reduced type corresponding to (I,ρ).

It is readily verified that direct products and direct sums commute with the taking of annihilators and products:

$$\text{(3.1)}\qquad \mathrm{Ann}\left(\square_{j\in J} A_j\right) = \square_{j\in J}\,\mathrm{Ann}(A_j) \qquad \text{and} \qquad \pi\left(\square_{j\in J} A_j\right) = \square_{j\in J}\pi(A_j)\,,$$

where $\square$ is either a direct sum or a direct product.

3.12. **Nilpotency and annihilator series for distributive algebras.** Central series for groups are well known, and the analogous concept of an annihilator series for nilpotent rings is developed in [14, Section 1.3]. Here we extend the concept of annihilator series from rings to distributive algebras.

Definition 3.13. A finite sequence of ideals $(A_j)_{j=0}^m$, $m\ge 0$, in an (I,ρ)-algebra A is said to be a *partial annihilator series* if

$$A_0 \trianglerighteq A_1 \trianglerighteq \cdots A_m$$

and $A_{j-1}/A_j \le \mathrm{Ann}(A/A_j)$ for $1\le i\le m$; note that this last condition is an (I_0,ρ_0)-subalgebra condition, where (I_0,ρ_0) is the reduced type corresponding to (I,ρ). An *annihilator series (of length m)* is a partial annihilator series $(A_j)_{j=0}^m$ such that $A_0 = A$ and $A_m = 0$. A is *nilpotent* if it has an annihilator series, and the *exponent*, $\exp(A)$, is the smallest length m of an annihilator series of A. We write $\exp(A)=\infty$ if A is not nilpotent.

We will now define upper and lower annihilator series as the algebra analogue of upper and lower central series for groups, and we will see that these are indeed annihilator series in the case of nilpotent algebras, generalizing Kruse and Price's result for rings ([14, Theorem 1.3.1]).

Definition 3.14. The *upper annihilator series* $(\mathrm{Ann}_j(A))_{j=0}^\infty$ is defined by

$$\begin{aligned}
&\mathrm{Ann}_0(A) := 0\,,\\
&\mathrm{Ann}_j(A) := \{a\in A \mid\ \forall\, i\in I_0,\ \underline{x}\in A^{\times\rho(i)} :\\
&\qquad\qquad\qquad a\in \mathrm{CS}(\underline{x}) \Rightarrow g_i^A(\underline{x}) \in \mathrm{Ann}_{j-1}(A)\}\,,\quad j\in\mathbb{N}\,.
\end{aligned}$$

Equivalently, $\mathrm{Ann}_{j-1}(A)\le \mathrm{Ann}_j(A)$ and

$$\mathrm{Ann}_j(A)/\mathrm{Ann}_{j-1}(A) = \mathrm{Ann}(A/\mathrm{Ann}_{j-1}(A))\,.$$

Definition 3.15. The *lower annihilator series* $(\pi_j(A))_{j=0}^\infty$ is defined by $\pi_0(A) = A$ and, for all $j\in\mathbb{N}$, $\pi_j(A)$ is defined inductively as the subgroup generated by elements of the form $g_i^A(\underline{x})$, $i\in I_0$, where at least one of the coordinates x_k lies in $\pi_{j-1}(A)$.

Examining the above definitions, it is clear that $\mathrm{Ann}_1(A)=\mathrm{Ann}(A)$, $\pi_1(A) = \pi(A)$, and that $\mathrm{Ann}_j(A)$ and $\pi_j(A)$ are ideals for all $j\ge 0$. Note that if A is a PN ring, then $\pi_j(A) = A^{j+1}$, where A^{j+1} is as defined in Section 2.

Lemma 3.16. *If $(A_j)_{j=0}^m$ is an annihilator series of a distributive algebra A, then $\pi_j(A) \leq A_j \leq \mathrm{Ann}_{m-j}(A)$ for all $0 \leq j \leq m$.*

Proof. Let (I, ρ) be the type of A. By definition $\pi_0(A) = A_0 = A$, so suppose $\pi_{j-1}(A) \leq A_{j-1}$ for some $1 \leq j \leq m$. Suppose $a \in \pi_{j-1}(A)$, and so $a \in A_{j-1}$. Since $A_{j-1}/A_j \leq \mathrm{Ann}(A/A_j)$, we have $g_i^A(\underline{x}) \in A_j$ whenever $i \in I_0$ and $a \in \mathrm{CS}(\underline{x})$. Since $i \in I_0$ and $a \in \pi_{j-1}(A)$ are arbitrary, we deduce that $\pi_j(A) \leq A_j$, and so this containment follows iteratively for all $0 \leq j \leq m$.

As for the other containment, we have $A_m = \mathrm{Ann}_0(A) = 0$, so suppose that $A_{j+1} \leq \mathrm{Ann}_{m-j-1}(A)$ for some $0 \leq j < m$. Now $A_j/A_{j+1} \leq \mathrm{Ann}(A/A_{j+1})$, so if $a \in A_j$ and $i \in I_0$, then $g_i^A(\underline{x}) \in A_{j+1} \leq \mathrm{Ann}_{m-j-1}(A)$ whenever $a \in \mathrm{CS}(\underline{x})$. Since $i \in I_0$ is arbitrary, it follows that $a \in \mathrm{Ann}_{m-j}(A)$, and so $A_j \leq \mathrm{Ann}_{m-j}(A)$. Thus this containment follows iteratively for all $0 \leq j \leq m$. □

It is now easy to deduce the following theorem.

Theorem 3.17. *The following conditions are equivalent for a distributive algebra.*

(a) *There exists $m \geq 0$ such that $\pi_m(A) = 0$.*
(b) *There exists $m \geq 0$ such that $\mathrm{Ann}_m(A) = A$.*
(c) *A is nilpotent.*

Moreover if A is nilpotent then $\exp(A)$ is both the least integer m such that $\pi_m(A) = 0$ and the least integer m such that $\mathrm{Ann}_m(A) = A$.

Proof. Clearly initial segments $(\mathrm{Ann}_j(A))_{j=0}^m$ and $(\pi_j(A))_{j=0}^m$ of the upper and lower annihilator series of A are partial annihilator series (written in reverse order in the case of the upper annihilator series), so both (a) and (b) imply (c). The fact that $\pi_j(A) \leq A_j$ for any annihilator series $(A_j)_{j=0}^m$ shows that (c) implies (a), and that $\exp(A)$ is the least integer m such that $\pi_m(A) = 0$. The fact that $A_j \leq \mathrm{Ann}_{m-j}(A)$ for any annihilator series $(A_j)_{j=0}^m$ shows that (c) implies (b), and that $\exp(A)$ is the least integer m such that $\mathrm{Ann}_m(A) = A$. □

Nilpotency and the nilpotent exponent behave well under taking of subalgebras, quotients, direct products, and direct sums, but generally the lower annihilator series behaves better than the upper annihilator series as we now explain.

If B is a subalgebra of A, it is readily verified that $\pi_j(B) \subseteq \pi_j(A)$ for $j \geq 0$, and so $\exp(B) \leq \exp(A)$. By contrast, there is in general no relationship between $\mathrm{Ann}_j(A)$ and $\mathrm{Ann}_j(B)$. For instance, if A is a semisimple Lie algebra and B is a one-dimensional subalgebra of A, then $0 = \mathrm{Ann}_j(A) \leq \mathrm{Ann}_j(B) = B$ for all $j \in \mathbb{N}$. By contrast if B is a subalgebra of a null algebra A, then $B = \mathrm{Ann}_j(B) \leq \mathrm{Ann}_j(A) = A$ for all $j \in \mathbb{N}$.

It is straightforward to verify that the upper or lower annihilator series of a direct product (or direct sum) of (I, ρ)-algebras A_j is obtained by taking direct products (or direct sums) of the corresponding terms in the upper or lower annihilator series of A_j. It is also clear that if the algebras A_j are all nilpotent of exponent at most m, then the direct product and direct sum of these algebras are also nilpotent of exponent at most m.

Finally we consider quotients or, equivalently, homomorphic images. The lower annihilator series is *fully invariant*, meaning that $\phi(\pi_j(A)) \subseteq \pi_j(B)$ whenever $\phi : A \to B$ is a homomorphism and $j \in \mathbb{N}$. This can be established by a routine induction proof which we omit. It follows that homomorphisms preserve nilpotency and that $\exp(\phi(A)) \le \exp(A)$ whenever ϕ is a homomorphism.

By contrast, the upper annihilator series is not fully invariant. To see this, it suffices to show that $\mathrm{Ann}(A)$ is not fully invariant. Consider a distributive algebra A which contains a null subalgebra N and satisfies $\mathrm{Ann}(A) = 0$: for instance, A could be a semisimple Lie algebra and N a one-dimensional subalgebra. Let $B := A \oplus N$ and define the endomorphism $f : B \to B$ by $\phi(x \oplus y) = y \oplus 0$. Then $\mathrm{Ann}(B) = 0 \oplus N$, but $\phi(\mathrm{Ann}(B))$ has trivial intersection with $\mathrm{Ann}(B)$.

The upper annihilator series is however invariant under surjective homomorphisms, as we now show.

Proposition 3.18. *If $\phi : A \to B$ is a surjective homomorphism of distributive algebras, then $\phi(\mathrm{Ann}_j(A)) \subseteq \mathrm{Ann}_j(B)$ for all $j \ge 0$.*

Proof. We prove the result inductively. It trivially holds for $j = 0$, so suppose it holds for $j = k-1$, $k \in \mathbb{N}$, and that $a \in \mathrm{Ann}_k(A)$. Thus $g_i^A(\underline{x}) \in \mathrm{Ann}_{k-1}(A)$ whenever $i \in I_0$, $\underline{x} \in A^{\times\rho(i)}$, and $a \in \mathrm{CS}(\underline{x})$. We fix such an a and write $b = \phi(a)$. Applying the homomorphism property, we see that $g_i^B(\underline{y}) \in \phi(\mathrm{Ann}_{k-1}(A)) \subseteq \mathrm{Ann}_{k-1}(B)$ whenever $i \in I_0$, $\underline{y} \in (\phi(A))^{\times\rho(i)}$, and $b \in \mathrm{CS}(\underline{y})$. Since $\phi(A) = B$, it follows that $b \in \mathrm{Ann}_k(B)$, completing the proof of the inductive step. □

3.19. **Multilinear polynomials and varieties of rings.** Multilinear polynomials provide a link between rings and distributive algebras. A *(nonassociative noncommutative) multilinear monomial in the unknowns* $X_1, \dots, X_n$ is an element of the free magma in these unknowns, with each unknown occurring exactly once. Equivalently, such a monomial is a nonempty nonassociative word in these variables where we use each unknown once and parentheses indicate the order of "multiplication": thus $X_1(X_2X_3)$, $(X_1X_2)X_3$, and $(X_2X_3)X_1$ are distinct monomials. The *degree* of such a multilinear monomial is the number n of unknowns.

The class of *(nonassociative noncommutative) multilinear polynomials of degree n over a commutative unital ring S*, $\mathrm{ML}_n(S)$, is the free S-module with basis consisting of all multilinear monomials in the unknowns $X_1, \dots, X_n$, i.e. sums of multilinear monomials of degree n, each multiplied by an element of S. We will only consider the case where the base ring S is $\mathbb{Z}$, but $S = \mathbb{Z}_m$ for m being a prime power could be useful in other situations.

When n is small, we denote the unknowns as $X, Y, \dots$ rather than $X_1, X_2, \dots$. We use function-style notation, writing either f or $f(X_1, \dots, X_n)$, depending on the situation. For instance, $f(X, Y, Z) := X(YZ) - 2(YX)Z \in \mathrm{ML}_3(\mathbb{Z})$. A *bilinear polynomial over* $\mathbb{Z}$ is an element of $\mathrm{ML}_2(\mathbb{Z})$, so it has the form $f(X, Y) := aXY + bYX$ for some $a, b \in S$. We write $\mathrm{ML}(\mathbb{Z}) = \bigcup_{n=1}^{\infty} \mathrm{ML}_n(\mathbb{Z})$.

Each element of $f \in \mathrm{ML}_n(\mathbb{Z})$ naturally gives rise to a function on any given PN ring R; we denote this function by f^R. We say that a PN ring R *satisfies* $f \in \mathrm{ML}_n(\mathbb{Z})$ if $f^R(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$, and we call f a *law of* R.

In the universal algebra literature, a *variety* is a class of algebraic structures of a given type satisfying a certain set of identities or laws. We use the term in this sense but we restrict the allowable laws to those that are multilinear. A subtle but important point of our definition is that we take the set V of laws that determine a variety $\mathcal{V}$ to be part of the structure of $\mathcal{V}$: if two classes of rings (or PN rings) $\mathcal{V}_1$ and $\mathcal{V}_2$ are equal as classes, and both are varieties but given by different sets of laws V_1 and V_2, then we do not consider $\mathcal{V}_1$ and $\mathcal{V}_2$ to be the same variety.

Suppose $V \subseteq \mathrm{ML}(\mathbb{Z})$. The *variety of rings determined by* V is the collection $\mathcal{V}$ of all rings R having every $f \in V$ as a law. Similarly we can define a *variety of PN rings determined by* V. For brevity, we speak of *associative varieties* and *PN varieties* instead of varieties of rings or of PN rings, respectively, and all varieties are determined by some $V \subseteq \mathrm{ML}(\mathbb{Z})$. A *variety* $\mathcal{V}$ means either an associative or a PN variety, and any element f of the determining set V is called a *law* of $\mathcal{V}$.

If $n \in \mathbb{N}$ is the maximal degree of a law of $\mathcal{V}$, we call V a *degree n variety.* We are especially interested in three fundamental degree two varieties that are each given by a single bilinear law: the variety of *null rings* $\mathcal{V}_{\mathrm{n}}$ is the collection of rings R with the law $f(X,Y) = XY$, while the varieties of *commutative* and *anticommutative rings* $\mathcal{V}_{\mathrm{c}}$ and $\mathcal{V}_{\mathrm{ac}}$ are the collections of rings with laws $f(X,Y) = XY - YX$ or $f(X,Y) = XY + YX$, respectively. The corresponding PN varieties are denoted $\mathcal{V}_{\mathrm{n,pn}}$, $\mathcal{V}_{\mathrm{c,pn}}$, and $\mathcal{V}_{\mathrm{ac,pn}}$, respectively; of course $\mathcal{V}_{\mathrm{n,pn}} = \mathcal{V}_{\mathrm{n}}$.

We also define two *trivial varieties*, both of which contain only the trivial ring: $\mathcal{V}_0$ is determined by the single law $f_1(X) := X$, while $\mathcal{V}_0'$ is determined by two laws, $f_1(X) := X$ and $f_2(X,Y) := XY$.

3.20. **Varieties and distributive algebras.** Suppose $\mathcal{V}$ is a variety determined by V. Let $I := V$ and $\rho(f) := \deg(f)$ for all $f \in V$. The *standard construction* of a distributive algebra from a PN ring R relative to $\mathcal{V}$ is to discard the original multiplication operation on R and replace it by the operations $g_f := f^R$, $f \in I$; we write $R_{\mathcal{V}}$ for the resulting (I,ρ)-algebra.

Given a ring or PN ring R, and a variety $\mathcal{V}$ determined by $V \subseteq \mathrm{ML}(\mathbb{Z})$, we define the *verbal subgroup* $V(R)$ to be the additive subgroup of R generated by all elements $f^R(x_1,\dots,x_n) \in R$, where $f \in V$ has degree $n = n(f)$.

We define the *marginal subgroup* $V^*(R)$ to be the set of all $x \in R$ such that $f^R(x_1,\dots,x_n) = 0$ whenever $f \in V$ and $x_1,\dots,x_n \in R$, with at least one of these elements equal to x.

It is clear that we have the set equations $V(R) = \pi(R_{\mathcal{V}})$ and $V^*(R) = \mathrm{Ann}(R_{\mathcal{V}})$. But, since we view $V(R)$ and $V^*(R)$ as being associated with the ring R rather than the algebra $R_{\mathcal{V}}$, there are differences between their closure properties: $V(R)$ and $V^*(R)$ are in general merely additive subgroups in R, whereas $\pi(R_{\mathcal{V}})$ and $\mathrm{Ann}(R_{\mathcal{V}})$ are ideals in the (I,ρ)-algebra $R_{\mathcal{V}}$.

The *lower marginal series* $(V_j(R))_{j=0}^\infty$ of R is the series of additive subgroups of R given by $V_j(R) := (\pi_j(A),+)$, $j \ge 0$. The *upper marginal series* $(V_j^*(R))_{j=0}^\infty$ of R is the series of additive subgroups of R given by $V_j^*(R) := (\mathrm{Ann}_j(A),+)$, $j \ge 0$. We say that R is *$\mathcal{V}$-nilpotent* if $R_{\mathcal{V}}$ is nilpotent, which is equivalent to $V_j(R) = 0$ for some $j \in \mathbb{N}$, and to $V_j^*(R) = (R,+)$ for some $j \in \mathbb{N}$.

Let us pause to consider the verbal and marginal subgroups for the trivial varieties and the three fundamental degree two varieties mentioned in §3.19.

(a) If $\mathcal{V} := \mathcal{V}_0$ or $V := \mathcal{V}_0'$, then $V(R) = R$ and $V^*(R) = 0$.
(b) If $\mathcal{V} := \mathcal{V}_{\mathrm{n}}$, then $V(R) = R^2$ and $V^*(R) = \mathrm{Ann}(R)$.
(c) If $\mathcal{V} := \mathcal{V}_{\mathrm{c}}$, then $V(R) = [R,R]$ and $V^*(R) = Z(R)$.
(d) If $\mathcal{V} := \mathcal{V}_{\mathrm{ac}}$, then $V(R) = \langle R,R\rangle$ and $V^*(R) = AZ(R)$.

Let us now consider the closure properties of $V(R)$ and $V^*(R)$ in the above cases. For $\mathcal{V} := \mathcal{V}_0$ or $V := \mathcal{V}_0'$, $V(R)$ and $V^*(R)$ are trivially ideals. For $\mathcal{V} = \mathcal{V}_{\mathrm{n}}$, both $V(R)$ and $V^*(R)$ are ideals in R; of course this must be so because $R = R_{\mathcal{V}}$ in this case. For $\mathcal{V} = \mathcal{V}_{\mathrm{c}}$, $V^*(R)$ is a subring but not necessarily an ideal, and $V(R)$ may fail even to be a subring. For $\mathcal{V} = \mathcal{V}_{\mathrm{n}}$, both $V^*(A)$ and $V(R)$ may fail to be subrings.

4. Isoclinism and isologism

We now develop a theory of isoclinism for distributive algebras. We will relate this theory to PN rings via the concept of isologism. Some earlier notions of isoclinism for some classes of PN rings in [14], [15], and [3] can be formulated in terms of these notions of isoclinism and isologism, as we will see in §4.22.

Although our terminology is inspired by analogous concepts in group theory, and although there are some echoes of group theoretic results in the theory that we develop, there are also some important differences. Perhaps the most important is that unlike group isoclinism, we will see that algebra isoclinism does not preserve nilpotency. But, rather than being a deficiency, this failure is a key feature of the theory and will be central to our proof of Theorem 1.1.

Another difference involves the relationship between isoclinism and isologism. Group isoclinism is a single equivalence relation on the class of all groups, while isologism is a family of such equivalence relations—one for each group variety—and isoclinism is just isologism with respect to a particular group variety. The relationship between isoclinism and isologism in this paper is quite different than this. Although isoclinism is a single concept for distributive algebras, each *type* of algebra gives rise to its own isoclinism. Isologism is a concept on the class of (PN) rings, and there is a distinct isologism for each appropriately defined variety of (PN) rings. Associated with each isologism is a map from the class of (PN) rings to the class of distributive algebras of a particular type, and a pair of (PN) rings are then isologic if the associated pair of distributive algebras are isoclinic.

Definition 4.1. An isoclinism from one (I, ρ)-algebra A to another one B consists of a pair of additive group isomorphisms $\phi : A/\operatorname{Ann}(A) \to B/\operatorname{Ann}(B)$ and $\psi : \pi(A) \to \pi(B)$ such that if $i \in I_0$ and $\phi(x_j + \operatorname{Ann}(A)) = y_j + \operatorname{Ann}(B)$, $j = 1, \dots, \rho(i)$, then $\psi(g_i^A(\underline{x})) = g_i^B(\underline{y})$. As usual, I_0 is the reduced index set.

We next define some functions that will allow us to give an alternative definition of the above isoclinism (ϕ, ψ). As mentioned in Remark 3.10, the operation g_i^R for $R = A, B$ gives rise to a multilinear map

$$\tilde{g}_i^R : (R/\operatorname{Ann}(R))^{\times n} \to \pi(R; i)\,,$$

and hence, via the universal property of tensor products, we get a surjective homomorphism

$$g_i^{R;\otimes} : (R/\operatorname{Ann}(R))^{\otimes n} \to \pi(R; i)\,,$$

The isomorphism $\phi : A/\operatorname{Ann}(A) \to B/\operatorname{Ann}(B)$ induces an isomorphism

$$\phi^{\otimes n} : (A/\operatorname{Ann}(A))^{\otimes n} \to (B/\operatorname{Ann}(B))^{\otimes n}\,.$$

Finally, we define ψ_i to be $\psi|_{\pi(A,i)}$.

With these newly defined maps, it follows that A and B are isoclinic via (ϕ, ψ) if and only if Figure 1 is a commutative diagram for each $i \in I_0$ and $n := \rho(i)$. The one part of this equivalence that is not immediately obvious is the fact that $\psi_i : \pi(A; i) \to \pi(B; i)$ is surjective, but this follows from the commutativity of the diagram and the surjectivity of the other three maps.

In particular, we make the following observation.

$$\begin{tikzcd}
(A/\operatorname{Ann}(A))^{\otimes n} \arrow[r, "\phi^{\otimes n}", "\cong"'] \arrow[d, two heads, "g_i^{A;\otimes}"'] & (B/\operatorname{Ann}(B))^{\otimes n} \arrow[d, two heads, "g_i^{B;\otimes}"] \\
\pi(A;i) \arrow[r, "\psi_i", "\cong"'] & \pi(B;i)
\end{tikzcd}$$

FIGURE 1. Isoclinism between (I, ρ)-algebras

Observation 4.2. If A and B are isoclinic, then $\pi(A;i)$ is isomorphic to $\pi(B;i)$.

The following result establishes some fundamental properties of isoclinism.

Theorem 4.3.

(a) *Isoclinism is an equivalence relation on the class of distributive algebras of any given type.*
(b) *All null algebras of a given type are isoclinic.*
(c) *If, for all j in a nonempty index set J, (ϕ_j, ψ_j) is an isoclinism from one (I, ϕ)-algebra A_j to another one B_j, then $\prod_{j\in J} A_j$ is isoclinic to $\prod_{j\in J} B_j$, and $\bigoplus_{j\in J} A_j$ is isoclinic to $\bigoplus_{j\in J} B_j$.*
(d) *Isomorphic algebras are isoclinic.*

Proof. Parts (a) and (b) are readily verified. As for (c), we write $A := \prod_{j\in J} A_j$ and $B := \prod_{j\in J} B_j$. It follows from (3.1) that there are product maps $\phi : A/\operatorname{Ann}(A) \to B/\operatorname{Ann}(B)$ and $\psi : \pi(A) \to \pi(B)$ defined in a coordinate-wise manner from the isoclinism maps for the pairs (A_j, B_j), and it is readily verified that (ϕ, ψ) is the desired isoclinism. The corresponding isoclinism for direct sums is obtained by restriction of ϕ and ψ.

The proof of (d) is fairly routine, but we include it for completeness. Let A, B be algebras of type (I, ρ), with (I_0, ρ_0) being the reduced type. We write 0 for both of the elements 0^A and 0^B and, fixing an arbitrary $i \in I_0$, we write $n := \rho(i)$. Let $\Phi : A \to B$ be an isomorphism, and let us write $\underline{\Phi} : A^{\times n} \to B^{\times n}$ for the cartesian product of n copies of Φ.

Proposition 3.18 implies that $\Phi(\operatorname{Ann}(A)) \subseteq \operatorname{Ann}(B)$. Thus $\phi(x+\operatorname{Ann}(A)) := \Phi(x) + \operatorname{Ann}(B)$, $x \in A$, gives a well-defined map $\phi : A/\operatorname{Ann}(A) \to B/\operatorname{Ann}(B)$.

It is clear that ϕ is a surjective group homomorphism, and we now show that it is injective. Suppose $\phi(x+\operatorname{Ann}(A)) = \phi(y+\operatorname{Ann}(A))$, and so $\Phi(z) \in \operatorname{Ann}(B)$, where $z := y - x$. Now let $\underline{x}, \underline{z} \in A^{\times n}$, with $z_j = z$ for some $1 \le j \le n$, and $z_k = 0$ for all $1 \le k \le n$, $k \ne j$. Let $\underline{w} \in A^{\times n}$ and $\underline{w}' \in B^{\times n}$ be such that $w_j = z_j$, $w_k = x_k$ for all other $1 \le k \le n$, and $w'_k = \Phi(w_k)$ for all $1 \le k \le n$. Then

$$\Phi(g_i^A(\underline{x}+\underline{z})) = \Phi(g_i^A(\underline{x})) + \Phi(g_i^A(\underline{w})) = \Phi(g_i^A(\underline{x})) + g_i^B(\underline{w}') = \Phi(g_i^A(\underline{x}))\,.$$

since $w'_j = \Phi(z_j) \in \operatorname{Ann}(B)$. Since Φ is injective, it follows that $g_i^A(\underline{x}) = g_i^A(\underline{x}+\underline{z})$, and so $z \in \operatorname{Ann}_i(A)$. Since $i \in I_0$ is arbitrary, we deduce that $z \in \operatorname{Ann}(A)$, and that ϕ is injective as desired.

We next define ψ to be $\Phi|_{\pi(A)}$. Clearly $\psi : \pi(B) \to B$ is an injective group homomorphism. Since $\Phi(g_i^A(\underline{x})) = g_i^B(\underline{\Phi}(\underline{x}))$ for all $\underline{x} \in A^{\times n}$, it follows that

$\psi(\pi(A)) \le \pi(B)$. On the other hand if $\underline{x}' \in B^{\times n}$, then the surjectivity of Φ implies that there exists $\underline{x} \in A^{\times n}$ such that $\underline{x}' = \underline{\Phi}(\underline{x})$, and now $\Phi(g_i^A(\underline{x})) = g_i^B(\underline{x}')$. Since $\pi(B)$ is generated by elements of the form $g_i^B(\underline{x}')$, we deduce that $\psi : \pi(A) \to \pi(B)$ is a surjection.

The defining identity for (ϕ, ψ) being an isoclinism from A to B now follows immediately from the isomorphism property of Φ. □

As discussed in §3.20, the standard construction associates a distributive algebra with each PN ring in a way that depends on a given variety $\mathcal{V}$. We now use this association to define *$\mathcal{V}$-isologism*.

Definition 4.4. Suppose $\mathcal{V}$ is a variety of rings (or PN rings) determined by $V \subseteq \mathrm{ML}(\mathbb{Z})$. A pair of rings (or PN rings), R and S, are said to be *$\mathcal{V}$-isologic* if for every $f \in V$, there are additive group isomorphisms $\phi : R/V^*(R) \to S/V^*(S)$ and $\psi : V(R) \to V(R)$ such that if $n = \deg(f)$ and $\phi(x_i + V^*(R)) = y_i + V^*(S)$, $i = 1, \dots, n$, then $\psi(f^R(x_1, \dots, x_n)) = f^S(y_1, \dots, y_n)$.

Equivalently, PN rings R, S are $\mathcal{V}$-isologic precisely when $R_{\mathcal{V}}$ and $S_{\mathcal{V}}$ are isoclinic.

We call an isoclinism equivalence class of (I, ρ)-algebras an *isoclinism family*, and we call an equivalence class of rings (or PN rings) with respect to $\mathcal{V}$-isologism a *$\mathcal{V}$-family*.

The proofs of the two theorems in the introduction will each employ only a single isoclinism (that of [2]-algebras), but they will use a different isologism for each symbol f.

4.5. **Isoclinism and isomorphism.** By Theorem 4.3, isomorphic algebras are isoclinic. We now look at the reverse implication. It is easy to give examples of isoclinic algebras that are non-isomorphic. Indeed we can take two (I, ρ)-algebras A and A', where A' is a null algebra of cardinality greater than 1. Then the direct sum $B := A \oplus A'$ is isoclinic to A by Theorem 4.3, but B and A are not even of the same cardinality.

Thus if we wish to find conditions under which isoclinic algebras A and B are necessarily isomorphic, it seems reasonable to include the assumptions $\mathrm{Ann}(A) = 0$ and $\mathrm{Ann}(B) = 0$ among those conditions.

We first show that these conditions alone are not sufficient, even in the context of classical algebras over a field.

Proposition 4.6. *There exist isoclinic but non-isomorphic classical algebras A and B over a field F such that* $\mathrm{Ann}(A)$ *and* $\mathrm{Ann}(B)$ *are both trivial.*

Proof. Let $A := F[X]$, the polynomial ring over a field F, and $B := XA$. Then $\mathrm{Ann}(A) = \mathrm{Ann}(B) = 0$ and there is a unique isomorphism ϕ from $(A, +)$ to $(B, +)$ taking X^j to X^{j+1} for all $j \ge 0$. Also $\pi(A) = A^2 = A$ and $\pi(B) = B^2 = X^2A$, and there is a unique isomorphism ψ from $(A^2, +)$ to $(B^2, +)$ taking X^j to X^{j+2} for all $j \ge 0$. It is readily verified that (ϕ, ψ) is an isoclinism from A to B, but A and B are non-isomorphic rings since A is unital and B is non-unital. □

We do not know of any pair of finite order isoclinic but non-isomorphic rings that have trivial annihilators, or any pair of isoclinic but non-isomorphic rings A and B that have trivial annihilators and satisfy $A^2 = A$, $B^2 = B$. However dropping associativity makes it easy to give examples of this type.

Proposition 4.7. *There exist two-dimensional classical PN algebras A and B over any field F such that*

(a) $\mathrm{Ann}(A)$ *and* $\mathrm{Ann}(B)$ *are both trivial.*
(b) $A^2 = A$ *and* $B^2 = B$.
(c) A *is associative.*
(d) A *is isoclinic to, but not isomorphic to,* B.

If instead A *and* B *are three-dimensional PN algebras over* F, *then we can arrange for (a)–(d) to be true, and for* A *and* B *to be unital.*

Proof. Let A be the (associative) classical algebra over F with basis $\mathcal{B} = \{a, b\}$ where $xy = x$ for all $x, y \in \mathcal{B}$. Let B be the nonassociative classical algebra over F such that $(B, +) = (A, +)$, $a^2 = ab = b$, and $ba = b^2 = a$. Parts (a), (b), and (c) are obviously true, and A and B are not isomorphic because A is associative and B is nonassociative (or because A has more idempotents than B if we pick F to be a finite field). However A is isoclinic to B via (ϕ, ψ), where ϕ is the identity map and ψ is defined by $\psi(a) = b$, $\psi(b) = a$.

The modification to 3-dimensional unital algebras consists basically of applying a Dorroh extension over F. Explicitly, we let $(A_1, +) := F \oplus A$, $(B_1, , +) := F \oplus B$. Define multiplication in A_1 by

$$(j \oplus a)(j' \oplus a') = jj' \oplus (ja' + j'a + aa')\,, \qquad j, j' \in F,\ a, a' \in A\,,$$

and define multiplication in B_1 analogously. It is readily verified that (a)–(d) remain true if we replace A by A_1 and B by B_1, and of course A_1 and B_1 are unital. □

The discussion so far of isoclinism versus isomorphism shows that isoclinism usually fails to imply isomorphism. We now switch to considering situations in which such an implication is possible for an isoclinism (ϕ, ψ) from one distributive algebra A to another B, beginning with the following simple result.

Proposition 4.8. *Isoclinic PN rings* A *and* B *are isomorphic if* A *and* B *have right unities* e_A *and* e_B, *respectively, with* $\phi(e_A) = e_B$.

Proof. The existence of one-sided unities in A and B immediately implies that $\mathrm{Ann}(A) = 0$, $\mathrm{Ann}(B) = 0$, $A^2 = A$, and $B^2 = B$. Thus ψ and ϕ are group isomorphisms from $(A, +)$ to $(B, +)$ and

$$\psi(x) = \psi(xe_A) = \phi(x)\phi(e_A) = \phi(x)e_B = \phi(x)\,, \qquad x \in A\,.$$

Thus $\psi = \phi$, and now the isoclinism property tells us that $\phi(xy) = \phi(x)\phi(y)$, so ϕ is the desired ring isomorphism. □

Proposition 4.8 is simple but not very satisfactory, since it is not hard to give examples of isoclinisms (ϕ, ψ) between (isomorphic) unital rings A and B where ϕ does not map the unity 1_A of A to the unity 1_B of B. For instance if $G := F(X)$ is the field of fractions of the polynomial ring $F[X]$ over some field F, then the equations $\phi(X^i) = X^{i+1}$ and $\psi(X^i) = X^{i+2}$, $i \in \mathbb{Z}$, can be extended uniquely to group homomorphisms ϕ, ψ on $(G, +)$, and it is readily verified that (ϕ, ψ) is an isoclinism from G to itself. It is therefore natural to ask if we can find isoclinic but non-isomorphic unital rings R and S. However the next theorem says in particular that this is not possible.

Suppose A is an (I, ρ)-algebra and let $i \in I_0$, $n := \rho(i)$, and $1 \le j \le n$. An (i, j)-*unity* for A is an element $e \in A$ such that $g_i^A(\underline{x}) = x_j$ whenever $x_k = e$ for all $k \ne j$.

Theorem 4.9. *Suppose* (ϕ, ψ) *is an isoclinism between distributive algebras* A *and* B. *Then* A *and* B *are isomorphic if either of the following conditions hold.*

(a) *There exists $i \in I_0$ and $1 \le j \le n := \rho(i)$ such that both A and B have (i,j)-unities e_A and e_B, respectively, and $\phi(e_A) = e_B$.*
(b) *A and B are unital rings.*

Proof. The proof of (a) is easy, and generalizes the argument in Proposition 4.8. First the existence of an (i,j)-unity for a distributive algebra R readily implies that $\mathrm{Ann}(R) = 0$ and $\pi(R) = R$, so ψ and ϕ are group isomorphisms from $(A,+)$ to $(B,+)$. Considering the isoclinism condition for $\underline{x} \in A^{\times\rho(i)}$ with $x_k = e_A$ for all $k \ne j$, we deduce that $\psi = \phi$, and so the isoclinism condition coincides with the isomorphism condition.

We now prove (b). Let 1_A and 1_B denote the unities of A and B, respectively. As before, ψ and ϕ are group isomorphisms from $(A,+)$ to $(B,+)$, $A^2 = \pi(A) = A$, and $B^2 = \pi(B) = B$. Let $e_B := \phi(1_A)$ and $e_A := \phi^{-1}(1_B)$. Now $xy = (xy)1_A = 1_A(xy)$, so the isoclinism property says that

$$\phi(x)\phi(y) = \phi(xy)e_B = e_B\phi(xy)\,, \qquad x, y \in A\,. \tag{4.1}$$

Products in A additively generate $A^2 = A$, and $\phi(A) = B$, so it follows from the second equation of (4.1) that $e_B \in Z(B)$.

Taking $x = y = e_A$ in (4.1), we see that e_B is invertible with $e_B^{-1} = \phi(e_A^2)$. Defining the group homomorphism $\eta : (A,+) \to (B,+)$ by $\eta(x) = e_B^{-1}\phi(x)$, it follows from (4.1) that

$$\eta(x)\eta(y) = e_B^{-2}\phi(x)\phi(y) = e_B^{-1}\phi(xy) = \eta(xy)\,.$$

Thus η is actually a ring isomorphism. □

Remark 4.10. The assumption $\phi(e_A) = e_B$ is essential in Theorem 4.9(a), since by Proposition 4.7 there are unital classical PN algebras that are isoclinic without being isomorphic.

We next see that by attaching the identity unitary operation to the structure of a general distributive algebra A, we can recast isomorphism as a special type of isoclinism.

Definition 4.11. Given a (I,ρ)-algebra A, we define another (I,ρ)-algebra A_{Id} as follows.

(a) A_{Id} has the same underlying set as A.
(b) $I_{\mathrm{Id}} = I \cup \{i_{\mathrm{Id}}\}$, where $i_{\mathrm{Id}} \notin I$.
(c) $\rho_{\mathrm{Id}} : I_{\mathrm{Id}} \to \mathbb{N}$ is defined by the equations $\rho_{\mathrm{Id}}|_I = \rho$ and $\rho(i_{\mathrm{Id}}) = 1$.
(d) $g_i^{A_{\mathrm{Id}}} = g_i^A$ for all $i \in I$.
(e) $g_{i_{\mathrm{Id}}}^A(x) = x$ for all $x \in A$.

Note that Id is naturally a functor on $\mathcal{Alg}_{\mathcal{D}}$, and that A and B are isomorphic if and only if A_{Id} and B_{Id} are isomorphic.

The importance of the Id functor is that every element in A_{Id} is an i_{Id}-unity (in a rather degenerate way, but this is sufficient).

Proposition 4.12. *Distributive algebras A and B are isomorphic if and only if A_{Id} and B_{Id} are isoclinic.*

Proof. Suppose A_{Id} and B_{Id} are isoclinic. The zero element is an i_{Id}-unity in both A_{Id} and B_{Id}, so Theorem 4.9(a) says that they are isomorphic. Conversely if A and B are isomorphic, then so are A_{Id} and B_{Id}, and hence they are isoclinic by Theorem 4.3(d). □

Remark 4.13. As mentioned previously, a null-type algebra is just an abelian group. It is also clear that all null-type algebras are isoclinic. However if we apply the Id functor to null-type algebras, we get the class of abelian groups "with added structure", and isoclinism on this class of structured abelian groups (which form a subclass of the [1]-algebras) corresponds to abelian group isomorphism.

Remark 4.14. The reason we defined two different trivial varieties of rings ($\mathcal{V}_0$ and $\mathcal{V}'_0$) in §3.19 can now be given: they allow us to show both ring isomorphisms, and additive group isomorphisms, between rings are special types of isologisms. First, it is clear that $\mathcal{V}_0$-isologism coincides with additive group isomorphism since in this case we have the single law $f_1(X) := X$, $V(R) = R$, $V^*(R) = 0$. Under the standard construction, the class of rings R give rise to a class of [1]-algebras which can also be obtained by applying the Id functor to the category of null-type algebras, so $\mathcal{V}_0$-isologism corresponds to isoclinism of structured abelian groups as discussed in Remark 4.13. In short, rings are $\mathcal{V}_0$-isologic if and only if their additive groups are isomorphic. On the other hand, the standard construction of distributive algebra of a ring R with respect to the variety $\mathcal{V}_2$ given by the single law $f_2(X, Y) := XY$ just returns the same ring. Appending the law $f_1(X) = X$ to $\mathcal{V}_2$ to obtain the trivial category $\mathcal{V}'_0$ corresponds under the standard construction to applying the Id functor to the category of rings. Thus by Proposition 4.12, rings are $\mathcal{V}'_0$-isologic if and only if they are (ring) isomorphic.

4.15. **Canonical form.** For group isoclinism, the notion of stem groups is important: these are groups G such that $Z(G) \leq [G, G]$. Equivalently for finite groups, a stem group is a group of minimal order in its isoclinism family. This notion does not however extend nicely to Z-isoclinism families of rings (see [3, Section 3]), and even requires modification in the context of general isologism families of groups [12, Section 8]. In its place for distributive algebras, it is useful to define a standard representative for each isoclinism family of algebras; we will say that this representative has canonical form.

Definition 4.16. A distributive algebra A has *canonical form* if:

(a) $(A, +)$ is the internal direct sum of subgroups A_1 and A_2.
(b) $\pi(A) = \operatorname{Ann}(A) = A_2$.

We call a canonical-form member of an isoclinism family a *canonical relative* of the other algebras in that family.

We now give an explicit construction for a canonical relative of any distributive algebra.

Definition 4.17. Given an (I, ρ)-algebra A, we define another (I, ρ)-algebra $\operatorname{Can}(A)$. First, $(\operatorname{Can}(A), +) := A_1 \oplus A_2$, where $A_1 = (A/\operatorname{Ann}(A), +)$ and $A_2 = (\pi(A), +)$. As for the other operations,

$$g_i^{\operatorname{Can}(A)}(x_1, \dots, x_n) := (0, g_i^{A/\operatorname{Ann}(A)}(u_1, \dots, u_n))$$

whenever $x_j = (u_j, v_j) \in A_1 \oplus A_2$, $i \in I$, $1 \leq j \leq n = \rho(i)$.

We next show that $\operatorname{Can}(A)$ is as promised a canonical relative of A, and that it is unique up to isomorphism. Additionally, we will see that canonical form algebras are more well-behaved than general algebras in that they are isoclinic if and only if they are isomorphic, a result that can be contrasted with Propositions 4.6 and 4.7.

Theorem 4.18.

(a) $\operatorname{Can}(A)$ *is a canonical relative of A whenever A is a distributive algebra.*
(b) *A pair of distributive algebras A and B are isoclinic if and only if* $\operatorname{Can}(A)$ *and* $\operatorname{Can}(B)$ *are isomorphic.*
(c) *Canonical form distributive algebras are isoclinic if and only if they are isomorphic.*
(d) *A canonical-form distributive algebra A is nilpotent of exponent at most 2.*
(e) *Nilpotency is not an isoclinism invariant.*

Proof. By definition, $A' := \operatorname{Can}(A)$ has the same type as A. It follows readily that $\pi(\operatorname{Can}(A)) = \operatorname{Ann}(\operatorname{Can}(A)) = 0 \oplus A_2$, and so $\operatorname{Can}(A)$ has canonical form. Identifying $A'/\operatorname{Ann}(A')$ with $A_1 \oplus 0$, we define $\phi : A/\operatorname{Ann}(A) \to A'/\operatorname{Ann}(A')$ by the identity $\phi(a+\operatorname{Ann}(A)) = (a+\operatorname{Ann}(A), 0)$, and $\psi : \pi(A) \to \pi(A')$ by the identity $\psi(a) = (0, a)$, we see that (ϕ, ψ) is an isoclinism from A to A'.

We next prove (b). Since isomorphic distributive algebras are isoclinic (Theorem 4.3(d)), and isoclinism is an equivalence relation, the fact that A and B are isoclinic if $\operatorname{Can}(A)$ and $\operatorname{Can}(B)$ are isomorphic follows from (a). Conversely, suppose that A and B are isoclinic via (ϕ, ψ). Thus $(\operatorname{Can}(A), +) := A_1 \oplus A_2$ and $(\operatorname{Can}(B), +) := B_1 \oplus B_2$, where A_1, A_2 are as in Definition 4.17 and B_1, B_2 are defined analogously. We write elements of $\operatorname{Can}(A)$ as $a = (a_1, a_2)$, and similarly $b = (b_1, b_2) \in \operatorname{Can}(B)$, where $a_i \in A_i$ and $b_i \in B_i$, $i = 1, 2$.

Now $\phi : A_1 \to B_1$ and $\psi : A_2 \to B_2$ are group isomorphisms, so we can define a group isomorphism $\Phi : (\operatorname{Can}(A), +) \to (\operatorname{Can}(B), +)$ by $\Phi(a) = (\phi(a_1), \psi(a_2))$. Suppose $a^1, \dots, a^n \in \operatorname{Can}(A)$, where $n = \rho(i)$, and $a^j = (a_1^j, a_2^j)$ for each index j. Let

$$x := \Phi(g_i^{\operatorname{Can}(A)}(a^1, \dots, a^n)) = (0, \psi(\tilde{g}_i^{A_1}(a_1^1, \dots, a_1^n)))$$

and

$$\begin{aligned} y &:= g_i^{\operatorname{Can}(B)}(\Phi(a^1), \dots, \Phi(a^n)) \\ &= g_i^{\operatorname{Can}(B)}((\phi(a_1^1), \psi(a_2^1)), \dots, (\phi(a_1^n), \psi(a_2^n))) \\ &= (0, \tilde{g}_i^{B_1}(\phi(a_1^1), \dots \phi(a_1^n))) . \end{aligned}$$

The fact that $x = y$ for all $a_1, \dots, a_1 \in \operatorname{Can}(A)$ follows from the definition of isoclinism. Since it is true for all $i \in I$, we have proved that $\operatorname{Can}(A)$ and $\operatorname{Can}(B)$ are isomorphic (as distributive algebras).

Part (c) follows immediately from (b) and the fact that $\operatorname{Can}(A)$ is isomorphic to $\operatorname{Can}(\operatorname{Can}(A))$, and (d) follows immediately from the fact that $\pi(A) = \operatorname{Ann}(A)$. Lastly, (e) follows from the fact that non-nilpotent (I, ρ)-algebras A exist for any type (I, ρ) with nonempty reduced index set I_0. For instance, we could take $(A, +)$ to be any nontrivial abelian group, and $g_i^A(\underline{x}) = x_1$ for $\underline{x} \in A^{\times \rho(i)}$ and all $i \in I_0$. $\square$

Remark 4.19. Proposition 4.12 tells us that a pair of distributive algebras A and B are isomorphic if A_{Id} and B_{Id} are isoclinic. However if A_{Id} is isoclinic to a general $(I_{\mathrm{Id}}, \rho_{\mathrm{Id}})$-algebra A', then A_{Id} and A' do not even have to be of the same cardinality. For instance, if A is an (I, ρ)-algebra with $1 < |A| < \infty$, then A_{Id} is isoclinic to $A' := \operatorname{Can}(A_{\mathrm{Id}})$ even though A_{Id} is non-nilpotent, A' is nilpotent, and $|A'| = |A|^2 > |A| = |A_{\mathrm{Id}}|$.

The next couple of corollaries of Theorem 4.18 give links between properties of canonical relatives of a distributive algebra A and properties of $A/\operatorname{Ann}(A)$. The first corollary is related to [3, Corollary 3.11] for Z-isoclinism, and [2, Proposition 2.10] for group isoclinism.

Corollary 4.20. *Suppose I is a finite index set. An (I,ρ)-algebra A is isoclinic to a finite algebra if and only if $A/\operatorname{Ann}(A)$ is finite.*

Proof. If A is isoclinic to a finite (I,ρ)-algebra B, then $A/\operatorname{Ann}(A) \cong B/\operatorname{Ann}(B)$, so certainly $A/\operatorname{Ann}(A)$ is finite. Conversely, suppose $A/\operatorname{Ann}(A)$ is finite. It follows by distributivity that for fixed $i \in I$, the product $g_i^A(\underline{x})$ depends only on the cosets $x_j + \operatorname{Ann}(A)$, $j := 1, \ldots, \rho(i)$. Thus A contains only finitely many product elements, and each product is of finite order in $(A,+)$; in fact its order divides $|A/\operatorname{Ann}(A)|$. Since $\pi(A)$ is additively generated by products, this is also of finite size. Every canonical-form algebra isoclinic to A has order $|A/\operatorname{Ann}(A)| \cdot |\pi(A)|$, so we are done. □

Corollary 4.21. *A distributive algebra A satisfies $m(A/\operatorname{Ann}(A)) = 0$ for a given integer m if and only if $m\operatorname{Can}(A) = 0$.*

Proof. Suppose an (I,ρ)-algebra A satisfies $m(A/\operatorname{Ann}(A)) = 0$. Thus $ma \in \operatorname{Ann}(A)$ for all $a \in A$ and so for all $i \in I_0$, $g_i^A(\underline{x}) = 0$ if $x_j \in mA$ for at least one index $1 \le j \le n := \rho(i)$. By distributivity, this means that $mg_i^A(\underline{y}) = 0$ for all $\underline{y} \in A^{\times n}$. Since $i \in I_0$ is arbitrary, we deduce that $m\pi(A) = 0$. But $\operatorname{Can}(A)$ is isomorphic as an additive group to $(A/\operatorname{Ann}(A)) \oplus \pi(A)$ so $m\operatorname{Can}(A) = 0$, as required. The converse direction follows immediately. □

4.22. **Comparisons involving isologism or isoclinism.** Finally in this section, we compare isologism and isoclinism with some other equivalence relations on classes of PN rings. First, isomorphism of PN rings is a special case of isologism: it corresponds to taking $\mathcal{V} = \mathcal{V}_0'$, as mentioned in Remark 4.14. Consider next the three types of isoclinism introduced in [3, Section 3]. Z-isoclinism, defined in [3] for rings, is simply $\mathcal{V}_{\mathrm{c}}$-isologism. G-isoclinism, defined in [3] for PN rings, is $\mathcal{V}_{\mathrm{n,pn}}$-isologism. (Although $\mathcal{V}_{\mathrm{n,pn}} = \mathcal{V}_{\mathrm{n}}$, we prefer to write $\mathcal{V}_{\mathrm{n,pn}}$ here to indicate that this is an isologism on a PN variety.)

Lastly, R-isoclinism was introduced in [3] to contrast with Z- and G-isoclinism, and it generalizes the types of isoclinism defined by Kruse and Price[3] [14] and Moneyhun [15]. R-isoclinism is related to $\mathcal{V}_{\mathrm{n,pn}}$-isologism but it is not the same: although both could be described as types of "isologism" with respect to the same variety, R-isoclinism is not an "isologism" in our sense of the word because the isoclinism maps ϕ, ψ are assumed to be ring isomorphisms rather than additive group isomorphisms. This makes R-isoclinism a finer equivalence relation than $\mathcal{V}_{\mathrm{n,pn}}$-isologism. However using Proposition 4.12, we can recover R-isoclinism from our notion of isoclinism by using quotient spaces and the Id functor: a pair of PN rings R and S are R-isoclinic if and only if $R/\operatorname{Ann}(R)$ and $S/\operatorname{Ann}(S)$ are isoclinic as $[1,2]$-algebras, where the attached operations are $g_1^T(x) := x$, and $g_2^T(x,y) := xy$ for $T = R/\operatorname{Ann}(R)$ and for $T = S/\operatorname{Ann}(S)$.

Nilpotency is not an isoclinism invariant (Theorem 4.18(e)), contrasting not only with the situation for groups, but also with the situation for R-isoclinism for rings (see [14, 3.1.5]). The fact that nilpotency is preserved by R-isoclinism

[3]Kruse and Price talk only of "families" and do not use the term "isoclinism".

but not by algebra isoclinism is a consequence of the fact that more of the algebraic structure is preserved by R-isoclinism than by algebra isoclinism.

One might therefore wonder why we do not assume that ϕ and ψ in our definition of isoclinism are algebra isomorphisms rather than group isomorphisms. One answer is that R-isoclinism is such a fine notion of equivalence relation that in some contexts it is little different from isomorphism: indeed two finite dimensional classical algebras over the same field are isoclinic if and only if one is isomorphic to the direct sum of the other one and a null algebra [14, Corollary 3.2.7]. Because less structure is preserved by algebra isoclinism, nothing like this is true there; see Propositions 4.6 and 4.7. Our notion is useful because it is both weak enough to allow every algebra to be isoclinic to an algebra of a rather simple canonical form, and strong enough that many interesting probabilistic functions are invariant with respect to suitable isoclinisms (as we see in the next section).

Our notion of canonical form (Definition 4.16) is close in spirit to the definition of Z-canonical form for rings in [3, Section 3], without being a true generalization of it: the earlier notion had an added condition defined in terms of the original ring multiplication, so it is a hybrid notion involving both the particular distributive algebra structure (in this case, a Lie ring) and the underlying ring structure. For that reason, Z-canonical form is not unique up to ring isomorphism, whereas our notion of canonical form notion is unique up to a distributive algebra isomorphism.

5. Applications

In this section we investigate various probabilistic and related functions on classes of finite cardinality distributive algebras A of a given type, and show that these functions are isoclinism invariants in this class, or some related class, of distributive algebras. Thus *a fortiori* they are also isomorphism invariants, but it is the isoclinism invariance property that will be more useful to us and will allow us to prove results such as those in the introduction.

Our initial task is to state a simple lemma that provides a useful isoclinism invariant. Using this and other isoclinism invariants, we then construct our probability functions. First though we need some notation.

Throughout this section A is a finite cardinality (I,ρ)-algebra, and $i \in I$ is a fixed index satisfying $n := \rho(i) \geq 2$, unless otherwise qualified. As before we write $\underline{x} = (x_1, \dots, x_n)$; $\underline{x}$ may be an element of $A^{\times n}$ or $(A/\operatorname{Ann}(A))^{\times n}$, depending on the situation. Let $\tilde{g}_i^A : (A/\operatorname{Ann}(A))^{\times n} \to \pi(A)$ be as in Remark 3.10. Whenever $\phi : A/\operatorname{Ann}(A) \to B/\operatorname{Ann}(B)$, we define

$$\underline{\phi} : (A/\operatorname{Ann}(A))^{\times n} \to (B/\operatorname{Ann}(B))^{\times n}$$

by $\underline{\phi}(\underline{x}) = \underline{y}$, where $\underline{y} = (y_1, \dots, y_n)$, and $\phi(x_j) = y_j$, $1 \leq j \leq n$. We similarly define the vector version

$$\underline{\psi} : (\pi(A))^{\times n} \to (\pi(B))^{\times n}$$

of $\psi : \pi(A) \to \pi(B)$. With this notation, the isoclinism condition is simply $\underline{\psi} \circ \tilde{g}_i^A = \tilde{g}_i^B \circ \underline{\phi}$.

5.1. **Invariant probability functions.** Lemma 3.2 of [3] says that one particular probability function (the commuting probability for rings) is an isoclinism invariant for one particular notion of isoclinism (Z-isoclinism). We now generalize that result.

Whenever A is a finite abelian group and $f : A^n \to A$ is a map for some $n \in \mathbb{N}$, we define

$$\Pr(A; f, n) := \frac{|\{\underline{a} \in A^{\times n} \ : \ f(\underline{a}) = 0\}|}{|A|^n}\,. \tag{5.1}$$

Lemma 5.2. *Suppose (ϕ, ψ) is an isoclinism from one finite (I, ρ)-algebra A to another B. Then $\Pr(A; g_i^A, n) = \Pr(B; g_i^B, n)$ for all $i \in I$ such that $n := \rho(i) > 0$.*

Proof. We can compute $\Pr(A; g_i^A, n)$ by counting only over cosets of $\operatorname{Ann}(A)$:

$$\Pr(A; g_i^A, n) := \frac{|\{\underline{x} \in (A/\operatorname{Ann}(A))^n \ : \ \tilde{g}_i^A(\underline{x}) = 0\}|}{|A/\operatorname{Ann}(A)|^n}\,.$$

But $\tilde{g}_i^A(\underline{x}) = 0$ if and only if $\underline{\psi}(\tilde{g}_i^A(\underline{x})) = \tilde{g}_i^B(\underline{\phi}(\underline{x})) = 0$. Since $\underline{\phi} : (A/\operatorname{Ann}(A))^n \to (B/\operatorname{Ann}(B))^n$ is an isomorphism, the lemma follows. ☐

The simple argument in the above lemma can be generalized. In particular, if replace g_i^R by $f^R : R^{\times m} \to R$ for $R = A, B$, with

$$f^R(x_1, \ldots, x_m) := g_i^R\left(\sum_{j=1}^m a_{1j}x_j, \ldots, \sum_{j=1}^m a_{nj}x_j\right)\,, \tag{5.2}$$

and we analogously define $\tilde{f}^R : (R/\operatorname{Ann}(R))^{\times m} \to R$, then we can prove the following lemma.

Lemma 5.3. *Suppose (ϕ, ψ) is an isoclinism from one finite (I, ρ)-algebra A to another B, and suppose f^A, f^B are defined as in (5.2), with $i \in I$, $n := \rho(i) > 0$, and $m \in \mathbb{N}$. Then $\Pr(A; f^A, m) = \Pr(B; f^B, m)$.*

Proof. As before, for $R = A$ and $R = B$, we have

$$\Pr(R; f^R, n) := \frac{|\{(x_1, \ldots, x_m) \in (R/\operatorname{Ann}(R))^m \ : \ \tilde{f}^R(x_1, \ldots, x_m) = 0\}|}{|R/\operatorname{Ann}(R)|^m}\,. \tag{5.3}$$

Now ψ is injective, so $\tilde{f}^A(x_1, \ldots, x_m) = 0$ if and only if $\underline{\psi}(\tilde{f}^A(x_1, \ldots, x_m)) = 0$. Moreover

$$\begin{aligned}
\underline{\psi}(\tilde{f}^A(x_1, \ldots, x_m)) &= \underline{\psi}\left(\tilde{g}_i^A\left(\sum_{j=1}^m a_{1j}x_j, \ldots, \sum_{j=1}^m a_{nj}x_j\right)\right) \\
&= \tilde{g}_i^B\left(\phi\left(\sum_{j=1}^m a_{1j}x_j\right), \ldots, \phi\left(\sum_{j=1}^m a_{nj}x_j\right)\right) \\
&= \tilde{g}_i^B\left(\sum_{j=1}^m a_{1j}\phi(x_j), \ldots, \sum_{j=1}^m a_{nj}\phi(x_j)\right) \\
&= \tilde{f}^B(\phi(x_1), \ldots, \phi(x_m))\,.
\end{aligned}$$

Thus $\tilde{f}^A(x_1,\dots,x_m) = 0$ if and only if $\tilde{f}^B(\phi(x_1),\dots,\phi(x_m)) = 0$. Since $\phi : A/\operatorname{Ann}(A) \to B/\operatorname{Ann}(B)$ is an isomorphism, the lemma now follows from (5.3). $\square$

The probability functions associated with isologism with respect to the varieties $\mathcal{V}_{\mathrm{n}}$, $\mathcal{V}_{\mathrm{c}}$, and $\mathcal{V}_{\mathrm{ac}}$ are covered by Lemma 5.2, but let us consider two other examples that might also be of interest.

Example 5.4. The condition $2x \in Z(R)$ comes up in many various commutativity results for rings: for instance if R satisfies an identity of the form $x^n - x \in Z(R)$ for some even integer n, it is readily deduced that $2x \in Z(R)$. Consequently, one might be interested in the associated probability function. Given a ring R, we associate a [2]-algebra A by attaching the operation

$$g_1^A(x,y,z) := 2xy - 2yx\,.$$

Then $\Pr(A; g_1^A, 2)$ is an isoclinism invariant for this associated [2]-algebra. Equivalently the probability that a general element in a ring R commutes with twice another element is an isologism invariant for the associative variety with the single law $f_1(X,Y) = 2XY - 2YX$.

Definition 5.5. An element x of a PN ring R is *dinilpotent* if $x^2 = 0$.

Example 5.6. A PN ring R is just a [2]-algebra. Taking $f^R(x) := x^2$, it follows that $\Pr(A; f^R, 2)$ is an isoclinism invariant of [2]-algebras, i.e. the proportion of dinilpotent elements in R is an isologism invariant for the variety $\mathcal{V}_{\mathrm{n}}$.

Example 5.7. We can combine invariant probability functions to get other invariant probability functions. For instance, if we were interested in investigating the spectrum of values of $\Pr_2(R) := \Pr_{\mathrm{c}}(R)\Pr_{\mathrm{ac}}(R)$ as R ranges over all PN rings R, it might be useful to use the fact that $\Pr_2(\cdot)$ is an isologism invariant for the variety with laws $f_1(X,Y) = XY - YX$ and $f_2(X,Y) = XY + YX$.

5.8. **Spectra and isologism.** Here we prove the results in the introduction and other related results involving spectra. We first prove the following stronger version of Theorem 1.1; note that, as a special case of Theorem 3.17, a PN ring R is said to be nilpotent of exponent at most n if $R^{n+1} = 0$. Also $\mathrm{ML}_n(\mathbb{Z})$ is as defined in §3.19.

Theorem 5.9. *Let C_0, $\mathcal{C}$, and $\mathcal{C}_{\mathrm{pn}}$ be the classes of all finite nilpotent rings of exponent at most 2, all finite rings, and all finite PN rings, respectively. Then $\mathfrak{S}_f(\mathcal{C}_1) = \mathfrak{S}_f(\mathcal{C})$ for all $f \in \mathrm{ML}_2(\mathbb{Z})$, and all classes $\mathcal{C}_1$ such that $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_{\mathrm{pn}}$.*

Proof. It is trivial that $\mathfrak{S}_f(\mathcal{C}_0) \subseteq \mathfrak{S}_f(\mathcal{C}_{\mathrm{pn}})$, so it suffices to show the converse inequality. To do this, we show that every PN ring R is $\mathcal{V}$-isologic to a ring S of exponent at most 2 with respect to some variety $\mathcal{V}$ such that $\mathcal{V}$-isologism has $\Pr_f$ as an invariant probability.

Let $\mathcal{V}_f$ be the PN variety whose single law is f, and let $V(R)$ and $V^*(R)$ be the associated verbal and marginal subgroups of R. It follows from Lemma 5.2 that $\Pr_f$ is a $\mathcal{V}_f$-isologism invariant.

We define a new PN ring S as follows: $(S,+) = R \oplus V(R)$, and multiplication is defined by $(x_1 \oplus x_2)(y_1 \oplus y_2) = 0 \oplus x_1y_1$ for $x_1, y_1 \in R$, $x_2, y_2 \in V(R)$. Written in terms of the direct sum decomposition of $(S,+)$, it is readily verified that $V(S) = 0 \oplus V(R)$ and $V^*(S) = V^*(R) \oplus V(R)$. Thus $S/V^*(S) = (R/V^*(R)) \oplus 0$. An isologism from R to S is given by (ϕ,ψ), where ϕ is the natural identification of $R/V^*(R)$ with the first summand of $S/V^*(S)$, and ψ is the natural identification of $V(R)$ with the second summand of $V(S)$. Thus $\mathrm{Pr}_f(R) = \mathrm{Pr}_f(S)$, and it is clear that S is nilpotent of exponent at most 2 (and thus associative), so we are done. $\square$

Remark 5.10. The proof of Theorem 5.9 also works for the functions considered in Examples 5.4–5.7: the associated spectra for finite nilpotent rings of exponent at most 2 each coincide with the corresponding spectrum for all finite PN rings. In particular, the dinilpotent spectra for rings and for PN rings coincides.

The dinilpotent condition $x^2 = 0$ is not so different in form from the idempotent condition $x^2 = x$, but idempotent proportion is not an isoclinism invariant. One might wonder if nevertheless the spectrum of possible idempotent proportions for PN rings might equal that for all rings. Since finite PN rings are direct sums of PN rings of prime power order, it suffices to consider the same question for PN p-rings and p-rings, where p is a prime. For "large" proportions there is no difference: the sets of possible idempotent proportions in the interval $[1/p, 1]$ for PN p-rings and for p-rings coincide ([4], [5]), and the sets of possible proportions in the interval $[2/p^2, 1]$ for unital p-rings and unital PN p-rings also coincide [6]. However when p is odd, there exists an idempotent proportion for (nonunital) PN p-rings exceeding $2/p^2$ (and at least two such proportions for $p > 3$) that is not an idempotent proportion for p-rings [7].

Proof of Theorem 1.2. Suppose R is a finite ring, and let $\mathcal{V}$ be the PN variety with law $f(X,Y) = aXY + bYX$. Let $R_{\mathcal{V}}$ be the standard construction of an associated (I,ρ)-algebra, as in §4.22, so that $R_{\mathcal{V}}$ is a PN ring, with multiplication $g^{R_{\mathcal{V}}}(x,y) := f^R(x,y)$. Let $S := \mathrm{Can}(R_{\mathcal{V}})$ with a single multiplication g^S. Like $R_{\mathcal{V}}$, S is a PN ring but, because S is nilpotent of exponent at most 2, it is actually a ring. Additionally $\mathrm{Pr}(S; g^S, 2) = \mathrm{Pr}(R_{\mathcal{V}}; g^{R_{\mathcal{V}}}, 2)$. But by construction, $\mathrm{Pr}(R_{\mathcal{V}}; g^{R_{\mathcal{V}}}, 2) = \mathrm{Pr}_f(R)$, while $\mathrm{Pr}(S; g^S, 2) = \mathrm{Pr}_{\mathrm{ann}}(S)$, so we are done. $\square$

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Department of Mathematics and Statistics, National University of Ireland Maynooth, Maynooth, Co. Kildare, Ireland.
E-mail address: stephen.buckley@maths.nuim.ie