ORLICZ-HARDY INEQUALITIES

Stephen M. Buckley and Pekka Koskela

Dedicated to Professor John L. Lewis on the occasion of his 60th birthday celebration.

Abstract. We relate Orlicz-Hardy inequalities on a bounded Euclidean domain to certain fatness conditions on the complement. In the case of certain log-scale distortions of L^n, this relationship is necessary and sufficient, thus extending results of Ancona, Lewis, and Wannebo.

0. Introduction

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and let $d(x) = \operatorname{dist}(x, \partial\Omega)$. We consider *integral Hardy inequalities*

$$\forall\, u \in C_0^\infty(\Omega): \qquad \int_\Omega \Psi\left(\frac{|u(x)|}{d(x)}\right) dx \le C \int_\Omega \Psi(|\nabla u(x)|)\, dx, \tag{0.1}$$

and *norm Hardy inequalities*

$$\forall\, u \in C_0^\infty(\Omega): \qquad \left\| \frac{|u(x)|}{d(x)} \right\|_{L^\Psi(\Omega)} \le C \|\, |\nabla u|\, \|_{L^\Psi(\Omega)} \tag{0.2}$$

where $\Psi : [0,\infty) \to [0,\infty)$ is any of a certain class of Orlicz functions with polynomial growth; see Section 1 for a definition of the Luxemburg norm $\|\cdot\|_{L^\Psi(\Omega)}$. Extending results of Ancona [A], Lewis [L], and Wannebo [W], who considered the case of power functions $\Psi(t) = t^p$, we relate the validity of such inequalities to certain fatness conditions on the complement of Ω. For

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1991 *Mathematics Subject Classification.* 46E35.

This paper was written while we were both at the University of Michigan for the Fred and Lois Gehring Special Year in Complex Analysis. We are both grateful for the hospitality of the Mathematics Department. The first author was partially supported by Enterprise Ireland.

Typeset by $\mathcal{AMS}$-TEX

power functions, it is clear that (0.1) and (0.2) are mutually equivalent, but this is not so in general. However, (0.1) implies (0.2); see Section 1.

In the classical case $\Psi(t) = t^p$, (0.1) holds for all $1 < p \le n$ on domains whose complement is locally uniformly p-fat, but it is only when $p = n$ that we get the equivalence of Hardy and local uniform fatness of the complement [L]. As a special case of our results, we now state an extension of this result to an interval in the log scale where we allow powers of $\log_+(t) = \log(t) \vee 1$ in the definition of Ψ.

Theorem 0.3. *Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded domain, and let $\Psi(t) = t^n \log_+^\alpha t$. Then*

(a) If $-1 \le \alpha \le n-1$, then (0.1), (0.2), and the local uniform n-fatness of $\mathbb{R}^n \setminus \Omega$ are all equivalent;

(b) If $\alpha > n-1$, (0.1) is equivalent to the local uniform n-fatness of $\mathbb{R}^n \setminus \Omega$;

(c) If $\alpha < -1$, then both (0.1) and (0.2) hold if $\mathbb{R}^n \setminus \Omega$ is locally uniformly n-fat.

Furthermore, the Hardy and fatness constants in all of the above implications and equivalences depend quantitatively only on each other, and on n, $\mathrm{dia}(\Omega)$, and α.

There are some significant differences between the classical case $\alpha = 0$ and the more general case above. First, it seems difficult to adapt Lewis' proof of the sufficiency of the fatness condition, so we adopt a different approach which is close to that of Wannebo [W]. Secondly, we shall see that the more tractable condition (0.1) is associated with a so-called *local uniform infimal Ψ-fatness* condition which is in general stronger than the natural definition of local uniform Ψ-fatness, but which coincides with local uniform p-fatness when $\Psi(t) = t^p \log_+^\alpha t$. We shall also need to investigate the more natural *local uniform Ψ-fatness* condition, as it arises in connection with (0.2).

In the power function case, local uniform p-fatness implies p-Hardy for all $p > 1$ (although local uniform p-fatness is a null condition when $p > n$). We extend this implication to a wide class of Young functions. As a special case, let us state such a result for logarithmically perturbed power functions.

Theorem 0.4. *Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded domain, and $\Psi(t) = t^p \log_+^\alpha t$, $1 < p < \infty$, $\alpha \in \mathbb{R}$. Suppose $\mathbb{R}^n \setminus \Omega$ is locally uniformly p-fat. Then Ω supports the Hardy inequality (0.1) with constant C dependent only on n, p, α, $\mathrm{dia}(\Omega)$, and the fatness constants r_0 and c of $\mathbb{R}^n \setminus \Omega$.*

The special case of Theorem 0.4 for $\Psi(t) = t^p$ was previously known: it was proved by Ancona [A] when $p = 2$, and Lewis [L] and Wannebo [W] for other values of p. Note also that for Lipschitz domains, much more precise results can be stated: Cianchi [C2] finds balance conditions between a pair of (not necessarily equal) Young functions that are necessary and sufficient for the validity of Hardy-type inequalities involving the associated Luxembourg norms.

We shall see that uniform Ψ-fatness for $\Psi(t) = t^n \log_+^\alpha t$ coincides with the well-understood uniform n-fatness condition if $\alpha \le n-1$, but is trivially satisfied if $\alpha > n-1$. Thus we have the following result.

Theorem 0.5. *Let $\Psi(t) = t^n \log_+^\alpha t$. Then singleton sets (and hence all sets) are locally uniformly Ψ-fat if $\alpha > n-1$. By contrast, if $\alpha \le n-1$, then singleton sets have zero Ψ-capacity, and local uniform Ψ-fatness coincides with local uniform n-fatness.*

In Section 2, we prove that a (local) uniform infimal Ψ-fatness condition often implies (0.1), and in Section 3, we establish the reverse implication in Orlicz classes near L^n. Finally in Section 4, we relate (infimal and non-infimal) Ψ-fatness to p-fatness in the special case $\Psi(t) = t^p \log_+^\alpha t$, thus allowing us to complete the proofs of the above theorems.

Let us close this introduction by noting that both (0.1) and (0.2) extend by the usual limiting argument to all u in $W_0^{1,\Psi}(\Omega)$, the $W^{1,\Psi}(\Omega)$-closure of $C_0^\infty(\Omega)$. Here, $W^{1,\Psi}(\Omega)$ is the Orlicz-Sobolev space with norm $\|u\|_{L^\Psi(\Omega)} + \|\,|\nabla u|\,\|_{L^\Psi(\Omega)}$, where $\|\cdot\|_{L^\Psi(\Omega)}$ denotes the usual Luxembourg norm on Ω with respect to Ψ. For more on Orlicz-Sobolev spaces, see [RR], [C1], and some of the references therein.

We wish to thank the referee for reading the paper carefully and spotting an error in Theorem 0.3.

1. Orlicz space preliminaries

We define an *Orlicz function* to be any convex homeomorphism $\Psi : [0,\infty) \to [0,\infty)$; thus an Orlicz function is essentially a Young function which is finite-valued and vanishes only at 0. If $\Omega \subset \mathbb{R}^n$ is a domain, the class $L^\Psi(\Omega)$ consists of all $f : \Omega \to \mathbb{R}$ such that $\int_\Omega \phi(c|f(x)|)\,dx < \infty$ for some $c > 0$. We define the *Luxemburg norm* $\|\cdot\|_{L^\Psi(\Omega)}$ by

$$\|f\|_{L^\Psi(\Omega)} = \inf\{t > 0 \mid \int_\Omega \phi(|f(x)|/t)\,dx \le 1\}. \tag{1.1}$$

This is a norm on $L^\Psi(\Omega)$ once we identify functions that agree almost everywhere; see theorem III.3.2.3 in [RR].

In this paper, we are particularly interested in the functions $\Psi(t) = t^p \log_+^\alpha t$, $1 < p < \infty$, $\alpha \in \mathbb{R}$. In this case, we define $\|\cdot\|_{L^\Psi(\Omega)}$ by (1.1) even though Ψ may fail to be both increasing and convex. Note, however, that we can always choose $K = K(p,\alpha) > 1$ such that such a function Ψ is increasing and convex on both of the intervals $[0,1]$ and $[K,\infty)$, and satisfies $\Psi(K) \geq K$. If we then define the function $\Psi_K : [0,\infty) \to [0,\infty)$ to coincide with Ψ on $[0,1] \cup [K,\infty)$, and to be defined by linear interpolation on $[1,K]$, then Ψ and Ψ_K are comparable, and so $\|\cdot\|_{L^\Psi(\Omega)}$ and $\|\cdot\|_{L^\Psi_K(\Omega)}$ are also comparable. Since we do not care about constants depending on p and ψ, we ignore the distinction between Ψ and Ψ_K and so can act as if ψ is an Orlicz function.

It turns out that (0.1) implies (0.2), with a comparable constant C, for any Orlicz function Ψ. To see this, let E denote the set of all $u \in C_0^\infty(\Omega)$ in the closed unit ball of $L^\Phi(\Omega)$, i.e. satisfying $\int_\Omega \Psi(|\nabla u(x)|)\,dx \leq 1$. Then

$$\int_\Omega \Psi\left(\frac{|u(x)|}{d(x)}\right) dx \leq C, \qquad u \in E.$$

But by convexity of Ψ and the fact that $\Psi(0) = 0$, it is clear that $\Psi(2t) \geq 2\Psi(t)$, and so

$$\int_\Omega \Psi\left(\frac{|u(x)|}{2^j d(x)}\right) dx \leq 2^{-j}C, \qquad u \in E, j \in \mathbb{N}.$$

We deduce (0.2) with associated constant 2^j as long as $j \in \mathbb{N}$ satisfies $2^j \geq C$.

2. Domains with fat complement support a Hardy inequality

Orlicz space capacities go back at least as far as the work of Aïssaoui and Benkirane [AB]; see also [K] and [AH]. If Ψ is an Orlicz function, then one can define the Ψ-capacity of a compact set $E \subset \Omega$ relative to an open set $\Omega \subset \mathbb{R}^n$ to be the infimum of the energy integrals $\int_\Omega \Psi(|\nabla u|)$ over all Lipschitz functions u that equal 1 on E, and 0 on $\partial\Omega$. One could analogously define a level-t capacity with respect to Ψ, where the energy integral is divided by $\Psi(t)$, and minimized over all functions that equal t on E, and 0 on $\partial\Omega$. For the L^p case, this is the same as the level-1 capacity, but for general Ψ it is not. Taking an infimum over all such level-t capacities gives an infimal capacity that we shall see is naturally associated with (0.1). Thus we have the following definitions.

Suppose Ψ is an Orlicz function and that E is a compact subset of an open set $\Omega \subsetneq \mathbb{R}^n$. We define the *level-$t$ Ψ-capacity* $\mathrm{cap}_\Psi^t(E;\Omega)$ and the *infimal Ψ-capacity* $\mathrm{cap}_\Psi^{\inf}(E;\Omega)$ by the equations

$$\begin{aligned}
\mathrm{cap}_\Psi^t(E;\Omega) &= \inf\left\{ \frac{\int_\Omega \Psi(|\nabla u(x)|)\,dx}{\Psi(t)} \,:\, u \in \mathrm{Lip}(\overline{\Omega}), u \ge t\chi_E, u|_{\partial\Omega} = 0 \right\}, \\
\mathrm{cap}_\Psi^{\inf}(E;\Omega) &= \inf_{t>0} \mathrm{cap}_\Psi^t(E;\Omega).
\end{aligned}$$

In particular, we write $\mathrm{cap}_\Psi(E;\Omega) = \mathrm{cap}_\Psi^1(E;\Omega)$. Writing $A_{x,r}(y) = (y-x)/r$, a set $E \subset \mathbb{R}^n$ is said to be *locally uniformly Ψ-fat* if there exist positive constants r_0, c, such that

$$\forall\, x_0 \in E,\ 0 < r < r_0 : \qquad \mathrm{cap}_\Psi(\overline{B}(0,1) \cap A_{x_0,r}(E); B(0,2)) \ge c,$$

and E is said to be *locally uniformly infimally Ψ-fat* if

$$\forall\, x_0 \in E,\ 0 < r < r_0 : \qquad \mathrm{cap}_\Psi^{\inf}(\overline{B}(0,1) \cap A_{x_0,r}(E); B(0,2)) \ge c.$$

If E is locally uniformly Ψ-fat for $\Psi(s) \equiv s^p$, we say that E is locally uniformly p-fat.

Let us say that an Orlicz function Ψ lies in the class $G(p,q,C)$, $1 \le p < q < \infty$, $C \ge 1$, if $\Psi(t^{1/p})/g(t) \in [1/C, C]$ and $\Psi(t^{1/q})/h(t) \in [1/C, C]$ for all $t > 0$, where g is a convex increasing function and h is a concave increasing function on $[0,\infty)$. These convexity assumptions constrain the growth rate of Ψ to be intermediate between $t \mapsto t^p$ and $t \mapsto t^q$. More precisely, since $f(st)/f(t) \ge s$ for any convex function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and the reverse inequality is true if f is concave, we deduce that

$$\forall\, s > 1,\ t > 0 : \qquad \Psi(st)/\Psi(t) \in [C^{-2}s^p, C^2 s^q]. \tag{2.1}$$

$$\forall\, s > 1,\ t > 0 : \qquad g(st)/g(t) \in [s, C^4 s^{q/p}]. \tag{2.2}$$

$$\forall\, s > 1,\ t > 0 : \qquad h(st)/h(t) \in [C^{-4} s^{p/q}, s]. \tag{2.3}$$

In particular, each of the functions Ψ, g, and h is *doubling* in the sense that its values at t and $2t$ are uniformly comparable for all $t > 0$.

For all $1 \le p_1 < p < p_2$ and $\alpha \in \mathbb{R}$, the function $\Psi(t) = t^p \log_+^\alpha t$ lies in $G(p_1, p_2, C)$ for some $C = C(p, \alpha, p_1, p_2)$; more precisely, this is true of the associated Orlicz function Φ_K, $K = K(\alpha, p)$, defined in Section 1. To see this, note that $\Psi(t^{1/p_1})$ is convex and $\Psi(t^{1/p_2})$ is concave on both $[0, 1/K')$ and (K', ∞) if $K' = K'(p_1, p_2, \alpha)$ is sufficiently large. Increasing K' if necessary, and redefining the functions on $[1/K', K']$ via linear interpolation gives the desired convex and concave functions.

We now state a sufficient condition for (0.1) which, after a little extra work in Section 4, will imply Theorem 0.4.

Theorem 2.4. *Suppose $\Omega \subset \mathbb{R}^n$, $n > 1$, is a bounded domain, and that $1 \le C_0$, $1 \le p < q$, and $q(n-p) < np$. If $\mathbb{R}^n \setminus \Omega$ is locally uniformly infimally Ψ-fat for some Orlicz function $\Psi \in G(p,q,C_0)$, then Ω supports the integral Hardy inequality (0.1) with constant C dependent only on n, p, q, $\mathrm{dia}(\Omega)$, and the fatness constants r_0 and c of $\mathbb{R}^n \setminus \Omega$.*

Our first step in proving Theorem 2.4 is the following lemma, which is of a well-known type that goes back to Maz'ya [Mz, Lemma 1]; see also [H, Lemma 2.1] and [KK, Lemma 3.1]). In this and later proofs, we write $A \lesssim B$ to mean that $A \le CB$, where C depends only on allowed parameters, and we write $A \approx B$ to mean $A \lesssim B \lesssim A$.

Lemma 2.5. *Let p, q, C_0, and Ψ be as in Theorem 2.4. Then there exists $C = C(n,p,q,C_0)$ such that*

$$\int_{B(0,1)} \Psi(|u|) \le \frac{C}{\mathrm{cap}_\Psi^{\inf}(Z; B(0,1))} \int_{B(0,2)} \Psi(|\nabla u|)$$

whenever $u \in \mathrm{Lip}(B(0,2))$ and $Z = \{x \in B(0,1) : u(x) = 0\}$.

Proof. Let us write $B = B(0,1)$, and let g, h be the functions in the $G(p,q,C_0)$ condition. Using Jensen's inequality for h^{-1}, we see that

$$\begin{aligned}
\int_B \Psi(|u-u_B|) &= h\left(h^{-1}\left(\int_B \Psi(|u-u_B|)\right)\right) \\
&\le h\left(\int_B h^{-1}(\Psi(|u-u_B|))\right) \\
&\lesssim \Psi\left(\left(\int_B |u-u_B|^q\right)^{1/q}\right). \qquad (2.6)
\end{aligned}$$

The last line follows from the doubling properties for h and Ψ and the consequent fact that $h^{-1}(\Psi(t)) \approx t^q$.

If $p < n$, then q is less than the Sobolev index $np/(n-p)$, so the classical Sobolev imbedding implies that

$$\left(\int_B |u-u_B|^q\right)^{1/q} \lesssim \left(\int_B |\nabla u|^p\right)^{1/p}. \qquad (2.7)$$

Since the (normalized) L^p norm of a function increases with p, and since the Sobolev index tends to infinity as p approaches n, it is clear that (2.7) also holds when $p \ge n$. Combining (2.6) and (2.7), we see that

$$\int_B \Psi(|u-u_B|) \lesssim g\left(\int_B |\nabla u|^p\right).$$

Again using Jensen's inequality, we see as before that

$$\int_B \Psi(|u-u_B|) \le \int_B g(|\nabla u|^p) \lesssim \int_B \Psi(|\nabla u|). \tag{2.8}$$

By doubling, there exists $C_1 = C_1(q, C_0)$ such that $\Psi(s+t) \le C_1(\Psi(s) + \Psi(t))$ for all $s, t \ge 0$. Thus

$$\int_B \Psi(|u|) \lesssim \Psi(|u_B|) + \int_B \Psi(|u-u_B|). \tag{2.9}$$

Simply because $Z \subset B$, we have $\operatorname{cap}_\Psi^{\inf}(Z; 2B) \lesssim 1$, and so by (2.8), we get the desired upper bound for the integral on the right-hand side of (2.9).

It remains to prove that $\Psi(|u_B|) \operatorname{cap}_\Psi^{\inf}(Z; 2B) \lesssim \int_{2B} \Psi(|\nabla u|)$. To see this, we take $v = |u - u_B|\eta$, where $\eta \in C_0^\infty(2B)$ has values in $[0,1]$, equals 1 throughout B, and satisfies $|\nabla \eta| \le 2$. Since v equals $|u_B|$ on Z and is compactly supported, we have

$$\Psi(|u_B|) \operatorname{cap}_\Psi^{\inf}(Z; 2B) \lesssim \int_{2B} \Psi(|\nabla v|).$$

Since $|\nabla v| \lesssim |\nabla u| + |u - u_B|$, we can use the variant of (2.8) with B replaced by $2B$, together with the doubling property of Ψ, to conclude that

$$\Psi(|u_B|) \operatorname{cap}_\Psi^{\inf}(Z; 2B) \lesssim \int_{2B} \Psi(|\nabla u|),$$

as desired. $\square$

By a change of variables, we get the following corollary.

Corollary 2.10. *Let p, q, C_0, and Ψ be as in Theorem 2.4. Then there exists $C = C(n, p, q, C_0)$ such that whenever $B = B(x_0, r)$, $x_0 \in \mathbb{R}^n$, $r > 0$, we have*

$$\int_B \Psi(|u|) \le \frac{C}{\operatorname{cap}_\Psi^{\inf}(Z; B(0,2))} \int_{2B} \Psi(r|\nabla u|)$$

for all $u \in \operatorname{Lip}(2B)$ and $Z = \{x \in B(0,1) : y \equiv rx + x_0 \in B, \, u(y) = 0\}$.

We are now ready to prove Theorem 2.4; our proof is inspired by the method used by Wannebo [W] to handle power functions.

Proof of Theorem 2.4. Both the Hardy inequality (0.1) and the uniform fatness condition are scale-invariant, so we assume without loss of generality that $\operatorname{dia}(\Omega) \le 1$. Let $\mathcal{W}$ be a Whitney cube decomposition of Ω (so that $\operatorname{dist}(Q, \partial\Omega)/\operatorname{dia}(Q) \in [1,4]$ for each $Q \in \mathcal{W}$), and for each $Q \in \mathcal{W}$, let x_Q be any point on $\partial\Omega$ that minimizes distance to Q. There is a constant $C_1 = C_1(n) < \infty$ such that $B_Q \equiv B(x_Q, r_Q)$ contains Q if $r_Q = C_1 \operatorname{dia}(Q)$. Thus also $2B_Q \subset KQ$, for some $K = K(n)$. Letting

$$\Omega_n = \{x \in \Omega : x \in Q \in \mathcal{W}, \operatorname{dia}(Q) \in (2^{-n}, 2^{-n+1}]\}, \quad n \in \mathbb{Z},$$
$$\widetilde{\Omega}_n = \bigcup_{m=n}^{\infty} \Omega_m, \quad n \in \mathbb{N}.$$

It follows that if $Q \in \mathcal{W}$ intersects Ω_n then $KQ \cap \Omega \subset \widetilde{\Omega}_{n-n_0}$, where $n_0 = 2 + \log_2(K)$. Note that the sets Ω_n (and so $\widetilde{\Omega}_n$) contain every Whitney cube that they intersect, that $\widetilde{\Omega}_1 = \Omega$, and that Ω_n is the empty set for all $n < 0$.

Suppose $u \in \operatorname{Lip}_0(\Omega)$, the subspace of $\operatorname{Lip}(\Omega)$ consisting of functions whose support is a compact subset of Ω. For arbitrary $0 < \alpha < 1$, we shall now derive some estimates which have constants of comparability dependent only on the parameters allowed in the statement of the theorem; however any dependence on α is explicitly given. We first apply Corollary 2.10 to the function $x \mapsto u(x)/r_Q^{1+\alpha}$, $0 < \alpha < 1$, and use uniform fatness of $\mathbb{R}^n \setminus \Omega$ to get that

$$\int_Q \Psi(d^{-1-\alpha}(x)|u(x)|)\,dx \approx \int_Q \Psi(r_Q^{-1-\alpha}|u(x)|) \lesssim \int_{KQ} \Psi(r_Q^{-\alpha}|\nabla u|),$$

assuming that $Q \in \mathcal{W}$ intersects Ω_n for some $n \ge n_1$, where $n_1 > n_0$ is large enough to ensure that r_Q is less than the uniform fatness parameter r_0. Since the cubes KQ, $Q \subset \Omega_n$, have uniformly bounded overlap, we deduce by summation over $Q \subset \Omega_n$ that for $n \ge n_1$,

$$\int_{\Omega_n} \Psi\left(\frac{|u(x)|}{d^{1+\alpha}(x)}\right) dx \lesssim \sum_{m=n-n_0}^{\infty} \int_{\Omega_m} \Psi(2^{n\alpha}|\nabla u|).$$

and so

$$\begin{aligned}\int_{\widetilde{\Omega}_{n_1}} \Psi\left(\frac{|u(x)|}{d^{1+\alpha}(x)}\right) dx &\lesssim \sum_{n=n_1}^{\infty} \sum_{m=n-n_0}^{\infty} \int_{\Omega_m} \Psi(2^{n\alpha}|\nabla u|) \\ &= \sum_{m=n_1-n_0}^{\infty} \int_{\Omega_m} \sum_{n=n_1}^{m+n_0} \Psi(2^{n\alpha}|\nabla u|) \\ &\lesssim \frac{1}{\alpha} \sum_{m=n_1-n_0}^{\infty} \int_{\Omega_m} \Psi(2^{m\alpha}|\nabla u|) \\ &\lesssim \frac{1}{\alpha} \int_{\Omega} \Psi(d^{-\alpha}(x)|\nabla u|) \end{aligned} \tag{2.11}$$

Note that the second inequality above follows by using (2.1) and summing a finite geometric series. Since $n_1 \lesssim 1$ and Ψ is doubling, it follows that

$$\begin{aligned}\int_{\Omega\setminus\widetilde{\Omega}_{n_1}} \Psi\left(\frac{|u(x)|}{d^{1+\alpha}(x)}\right) dx &\lesssim \int_{\Omega} \Psi\left(|u(x)|\right)\, dx \\ &\lesssim \int_{\Omega} \Psi\left(|\nabla u(x)|\right)\, dx \\ &\le \int_{\Omega} \Psi\left(d^{-\alpha}(x)|\nabla u(x)|\right)\, dx. \end{aligned} \tag{2.12}$$

The second inequality here follows in a similar manner to (2.8), except that we use the classical Sobolev imbedding for compactly supported functions in place of (2.7). Combining (2.11) and (2.12), we deduce that

$$\int_{\Omega} \Psi\left(\frac{|u(x)|}{d^{1+\alpha}(x)}\right) dx \lesssim \frac{1}{\alpha} \int_{\Omega} \Psi\left(d^{-\alpha}(x)|\nabla u(x)|\right)\, dx. \tag{2.13}$$

Since (2.13) holds for any $\mathrm{Lip}_0(\Omega)$ function, we may replace u by $v = ud^{\alpha}$ to deduce that

$$\int_{\Omega} \Psi\left(\frac{|u(x)|}{d(x)}\right) dx \approx \int_{\Omega} \Psi\left(\frac{|v(x)|}{d^{1+\alpha}(x)}\right) dx \lesssim \frac{1}{\alpha} \int_{\Omega} \Psi(d^{-\alpha}(x)|\nabla v|)\, dx. \tag{2.14}$$

But $|\nabla v| \lesssim d^{\alpha}|\nabla u| + \alpha d^{\alpha-1}|u|$, and so

$$\begin{aligned}\int_{\Omega} \Psi\left(\frac{|u|}{d}\right) &\le \frac{C_1}{\alpha} \int_{\Omega} \Psi(|\nabla u| + \alpha d^{-1}|u|) \\ &\le \frac{C_2}{\alpha} \left(\int_{\Omega} \Psi(|\nabla u|) + \int_{\Omega} \Psi\left(\frac{\alpha|u|}{d}\right)\right), \end{aligned} \tag{2.15}$$

where $C_1, C_2 \lesssim 1$ and $C_2 > 1$. But if we take $\alpha = (2C_2C_0^2)^{-1/(p-1)} \approx 1$, then (2.1) implies that

$$\frac{C_2}{\alpha}\int_\Omega \Psi\left(\frac{\alpha|u|}{d}\right) \le \frac{1}{2}\int_\Omega \Psi\left(\frac{|u|}{d}\right).$$

From this last estimate and (2.15) we get the desired conclusion. □

3. Domains supporting a Hardy inequality have fat complement

In this section, we state and prove a version of "Hardy implies uniform fatness" for Orlicz spaces near L^n. After a little extra work in the next section, this will imply parts of Theorem 0.3(a), (b). We begin by defining the concept of a quasilog, which replaces the power of a logarithm in Theorem 0.3.

A function $\phi : (0,\infty) \to (0,\infty)$ is said to be a *quasilog* and lie in the class $QL(K)$, $K \ge 1$, if $\phi(s)/\phi(t) \le K$ for all positive numbers s, t satisfying either $t^2 \wedge t^{1/2} \le s \le t^2 \vee t^{1/2}$, or $s, t \in [1/2, 2]$.

Note that the opposite inequality $\phi(t)/\phi(s) \le K$ follows whenever s, t satisfy the same conditions, and that any such function is *doubling* in the sense that $\phi(t)/\phi(s) \le K$ whenever $0 \le t/2 \le s \le 2t$.

If the function ϕ is Lipschitz and satisfies

$$\sup_{t>0}\int_{I_t} \frac{|\phi'(s)|}{\phi(s)} ds < C < \infty, \tag{3.1}$$

where I_t is the interval $[1/2, 2]$ for $1/2 \le t^2 \le 2$, and the interval with endpoints t and t^2 for all other $t > 0$, then it is clear that $\phi \in QL(\exp(C))$.

Taking $\phi_\alpha(t) = \log_+^\alpha(t)$, $t > e$, $\alpha \in \mathbb{R}$, we have $|\phi_\alpha'(t)|/\phi_\alpha(t) = |\alpha|/(t \log t)$, and the left-hand side of (3.1) is just $|\alpha| \log 2$. The case of $t \le \sqrt{e}$ is of course trivial for ϕ_α, and the intermediate case is easily handled, so it follows that $\phi_\alpha \in QL(2^{|\alpha|})$. More generally, it is not hard to check directly from the original definition that if ϕ is any finite product of powers of $\log_+$ and its iterates is a quasilog. Conversely, it is not hard to see that quasilogs cannot grow or decay faster than a bounded power of log, with the bound dependent only on the constant K.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n > 1$, be a bounded domain, and $1 \le K$. If the integral Hardy inequality (0.1) holds for some Orlicz function Ψ such that $\Psi(t) = t^n\phi(t)$ for a function $\phi \in QL(K)$ satisfying*

$$\int_1^\infty \frac{\phi(t)\,dt}{t} = \infty,$$

then $\mathbb{R}^n \setminus \Omega$ is locally uniformly infimally Ψ-fat. If instead the norm Hardy inequality (0.2) holds for such a Ψ, then $\mathbb{R}^n \setminus \Omega$ is locally uniformly Ψ-fat. In both cases, we can choose the uniform fatness constants to satisfy $r_0 = \mathrm{dia}(\Omega)$ and $c = c(C, n, \mathrm{dia}(\Omega), \Psi)$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus \Omega$. Suppose first that (0.1) holds but that

$$\mathrm{cap}_\Psi^{\inf}(\overline{B}(0,1) \cap A_{x_0,r}(\Omega^c); B(0,2)) < \epsilon$$

for some very small number $\epsilon > 0$; without loss of generality we assume that $r < 1$. Thus there exists a number $L > 0$ and a function $g \in C_0^\infty(B(0,2); [0,L])$ such that $g(x) = L$ for all $x \in \overline{B}(0,1) \cap A_{x_0,r}(\Omega^c)$, and $\int_{B(0,2)} \Psi(|\nabla g|) \le \epsilon \Psi(L)$.

Let us write $v = (L-g)\eta$, where $\eta \in C_0^\infty(B(0,1); [0,1])$ equals 1 throughout $B(0,1/2)$ and $\|\nabla \eta\|_\infty \le 3$. Since ϕ, and so Ψ, is doubling, we conclude that

$$\int_{B(0,2)} \Psi(|\nabla v|) \le \int_{B(0,2)} \Psi(3(L-g)) + \int_{B(0,2)} \Psi(|\nabla g|) \lesssim \Psi(L). \tag{3.3}$$

Writing $S = \{x \in B(0,1/2) : g(x) \ge L/2\}$, it follows from the Sobolev inequality and Jensen's inequality that

$$\frac{L|S|}{2} \le \int_{B(0,2)} g \lesssim \int_{B(0,2)} |\nabla g| \le \Psi^{-1}\left(\int_{B(0,2)} \Psi(|\nabla g|) \right).$$

Since Ψ is doubling, this last set of inequalities implies that

$$\Psi(L|S|) \lesssim \int \Psi(|\nabla g|) \le \epsilon \Psi(L).$$

Writing $c_n = |B(0,1)|/2^{n+1}$, it follows that $|S| \le \delta^n c_n$ for some number δ which tends to zero as ϵ tends to zero. By choosing ϵ sufficiently small, we may assume in particular that $\delta \le 1/8$.

Writing $u(x) = v((x-x_0)/r)$ on $\Omega \cap B(x_0,r)$, and $u(x) = 0$ for $x \in \Omega \setminus B(x_0,r)$, we see that $u \in C_0^\infty(\Omega)$. Writing

$$G = \{ t \in (0,1/2) : H^{n-1}(S \cap \partial B(0,t)) \le H^{n-1}(\partial B(0,t))/2 \},$$

the inequality $|S| \le c_n \delta^n$ implies that $|G| \ge 1/2 - \delta/2$. This last inequality implies that

$$L^n \int_\delta^{1/2} \frac{\phi(L/t)\, dt}{t} = \int_\delta^{1/2} t^{n-1} \Psi(L/t)\, dt \lesssim \int_G t^{n-1} \Psi(L/t)\, dt. \tag{3.4}$$

To see this, let j_0 be the greatest integer less than or equal to $\log_2(1/\delta) - 1$, and partition the interval $[\delta, 1/2]$ into subintervals of the form $[2^{j-1}\delta, 2^j\delta)$, $j = 1, \dots, j_0 - 1$, and $[2^{j_0-1}, 1/2]$. Since $|G| \geq 1/2 - \delta/2$, at least half the length of each of these intervals is contained in G. Using the doubling property of Ψ, we get the desired inequality. It follows that

$$
\begin{aligned}
L^n \int_\delta^{1/2} \frac{\phi(L/t)\,dt}{t} &\lesssim \int_{B(0,2)\setminus S} \Psi\left(\frac{v(y)}{|y|}\right) dy \\
&\leq \int_{B(0,2)} \Psi\left(\frac{v(y)}{|y|}\right) dy \\
&= r^{-n} \int_{\Omega \cap B(x_0,2r)} \Psi\left(\frac{r|u(x)|}{d(x)}\right) dx \\
&\lesssim r^{-n} \int_\Omega \Psi(r|\nabla u|) \\
&= \int_{B(0,2)} \Psi(|\nabla v|) \lesssim L^n \phi(L).
\end{aligned} \tag{3.5}
$$

Note that the first inequality above follows from (3.4), Fubini's theorem, the definition of G, and the fact that $v(x) \geq L/2$ for all $x \in B(0,1/2) \setminus S$, while the other two inequalities follow from (0.1) and (3.3), and the two equations are simple changes of variable. Writing

$$
I(L,\delta) = \frac{1}{\phi(L)} \int_\delta^{1/2} \frac{\phi(L/t)\,dt}{t},
$$

it follows from (3.5) that $I(L,\delta) < C_0$, for some $C_0 = C_0(C, n, \mathrm{dia}(\Omega), \Psi) > 1$. We get the desired contradiction if we can show that there exists $\delta > 0$ such that $I(L,\delta) \geq C_0$ for all $L > 0$ (since such δ can be chosen by taking $\epsilon > 0$ small enough).

Let $C = 4\exp(C_0 K)$. If $C^{-2} \leq L \leq C$, then by doubling of ϕ we have

$$
\begin{aligned}
I(L,\delta) = \frac{1}{\phi(L)} \int_\delta^{1/2} \frac{\phi(L/t)\,dt}{t} &\approx \int_\delta^{1/2} \frac{\phi(1/t)\,dt}{t} \\
&= \int_2^{1/\delta} \frac{\phi(s)\,ds}{s},
\end{aligned}
$$

and so $I(L,d) \geq C_0$ for some sufficiently small $\delta > 0$ which is independent of L.

On the other hand if L does not lie in the interval $[C^{-2},C]$, the $QL(K)$ condition implies that for $0<\delta<1/C$ we have

$$\begin{aligned} I(L,\delta) \geq \frac{1}{\phi(L)}\int_{1/C}^{1/2}\frac{\phi(L/t)\,dt}{t} &= \frac{1}{\phi(L)}\int_{2L}^{CL}\frac{\phi(s)\,ds}{s} \\ &\geq \frac{1}{K}\int_{2L}^{CL}\frac{ds}{s} = \log(C/2)/K = C_0. \end{aligned}$$

Again, $I(L,\delta)\geq C_0$ for some sufficiently small $\delta>0$ which is independent of L.

The proof for the norm Hardy inequality is similar but a little easier. Arguing by contradiction as before, we may assume that $g\in C_0^\infty(B(0,2);[0,1])$ is such that $g(x)=1$ for all $x\in\overline{B}(0,1)\cap A_{x_0,r}(\Omega^c)$, and $\int_{B(0,2)}\Psi(|\nabla g|)\leq\epsilon$. The argument then proceeds in the same manner as before (with L replaced by 1, and $\Psi(L)$ replaced by 1), until we deduce an analogue of (3.5). This analogue implies that $\int_\delta^1 t^{-1}\phi(1/t)\,dt$ is uniformly bounded over all $\delta>0$ which, by a change of variables, contradicts the unboundedness of $\int_1^\infty t^{-1}\phi(t)\,dt$. $\square$

4. Capacities related to $\Psi(s)=s^p\log_+^\alpha s$.

In this section, we examine in more detail capacities related to the Orlicz function $\Psi(s)=s^p\log_+^\alpha(s)$. We first prove Theorem 0.5, which states that Ψ-fatness is equivalent to n-fatness if $p=n$ and $\alpha\leq n-1$ (but not if α is larger than $n-1$). We then prove two lemmas which tell us that infimal Ψ-fatness is equivalent with p-fatness; unlike the non-infimal case, this works for all $\alpha\in\mathbb{R}$ (and all $p>1$).

Proof of Theorem 0.5. Applying the Orlicz version of Hölder's inequality [RR, p.58] to the following "representation formula" (for which, see [GT, Lemma 7.14])

$$\forall\, u\in \mathrm{Lip}_0(B(0,2)): \qquad |u(x)|\leq C_n\int_{B(0,2)}\frac{|\nabla u|\,dy}{|x-y|^{n-1}}, \tag{4.1}$$

we get

$$\forall\, u\in \mathrm{Lip}_0(B(0,2)): \qquad u(0)\leq 2C_n\|\,|\nabla u|\,\|_{L^\Psi(B(0,2))}\|\,|x|^{-n+1}\,\|_{L^{\widetilde{\Psi}}(B(0,2))}.$$

where $\widetilde{\Psi}(t)\approx t^{n/(n-1)}\log_+^{-\alpha/(n-1)}t$ is the conjugate Orlicz function to Ψ. Since

$$\|\,|x|^{-n+1}\,\|_{L^{\widetilde{\Psi}}(B(0,3))}<\infty$$

whenever $\alpha > n-1$, we get a lower bound for $\mathrm{cap}_\Psi(\{0\}; B(0,2))$. This implies that a singleton (and hence every set) is locally uniformly Ψ_α-fat if $\alpha > n-1$.

Suppose instead that $\alpha \le n-1$. If we take the test functions

$$u_\epsilon(x) = \begin{cases} \dfrac{\log_+ \log_+(1/|x|) - 1}{\log_+ \log_+(1/\epsilon) - 1}, & |x| \ge \epsilon, \\ 1, & |x| \le \epsilon, \end{cases}$$

and let ϵ tend to zero, we see that $\{0\}$ has Ψ_α-capacity zero (and so is certainly not locally uniformly Ψ_α-fat). By using similar test functions, we see that $\mathrm{cap}_\Psi(\overline{B}(0,r); B(0,2))$ tends to zero as r tends to zero. It follows that if E is locally uniformly Ψ-fat then every annulus $B(x,R) \setminus B(x,r)$ must contain a point of E whenever $x \in E$ and $R/r > C$, where C depends on n, α, and the fatness constant. This is the well-known *uniformly perfect* condition, and it is known to be equivalent to n-fatness [JV, Theorem 4.1]. The converse direction (from n-fat to Ψ-fat) follows from Jensen's inequality and the convexity near infinity of $t \mapsto \Psi_\alpha(t^{1/n})$ if $\alpha > 0$. If instead $\alpha < 0$, we first use Lewis's result [L, Theorem 1], that a locally uniformly n-fat set must be locally uniformly p-fat for some $p < n$, and then go from p-fat to Ψ-fat via Jensen's inequality and the convexity near infinity of $t \mapsto \Psi_\alpha(t^{1/p}) \approx t^{n/p} \log_+^\alpha(t)$. □

Using Jensen's inequality and [L, Theorem 1] as in the last paragraph of the above proof, we get the following partial analogue of Theorem 0.5 for general p.

Proposition 4.2. *If $\Psi(t) = t^p \log_+^\alpha t$, $p > 1$, $\alpha \le 0$, then local uniform Ψ-fatness coincides with local uniform p-fatness.*

We next relate infimal Ψ-fatness with p-fatness, considering the cases $\alpha \le 0$ and $\alpha > 0$ separately.

Lemma 4.3. *Let $\Psi(s) = s^p \log_+^\alpha(s)$ for some $\alpha \le 0$, $p > 1$. Then $\mathrm{cap}_\Psi^t(E;\Omega) \ge \mathrm{cap}_\Psi^e(E;\Omega)$ for every compact set E in an open set $\Omega \subsetneq \mathbb{R}^n$, and all $t > 0$. Thus $\mathrm{cap}_\Psi^{\inf}(E;\Omega) \ge c\, \mathrm{cap}_\Psi(E;\Omega)$ for some $c = c(\alpha) > 0$. In particular, a set is locally uniformly infimally Ψ-fat if and only if it is locally uniformly p-fat.*

Proof. Since $\mathrm{cap}_\Psi(E;\Omega) \approx \mathrm{cap}_\Psi^e(E;\Omega)$, the second statement follows readily from the first, and the third statement then follows from Proposition 4.2. Thus it suffices to prove the fist statement. Writing $A_t(E;\Omega)$ for the set of admissable test functions for $\operatorname{cap}_\Psi^t(E;\Omega)$ (i.e., Lipschitz functions which equal t on E and zero on $\partial\Omega$), we claim that for all $t>0$, and all $u \in A_e(E;\Omega)$,

$$\begin{aligned} \frac{\int_\Omega \Psi(|e^{-1}t\nabla u|)}{\Psi(t)} &= e^{-p}\int_\Omega |\nabla u|^p \log_+^\alpha(e^{-1}t|\nabla u|)\log_+^{-\alpha} t \\ &\geq e^{-p}\int_\Omega |\nabla u|^p \log_+^\alpha(|\nabla u|) \geq \operatorname{cap}_\Psi^e(E;\Omega). \end{aligned}$$

Since $A_t(E;\Omega) = \{e^{-1}tu : u \in A_e(E;\Omega)\}$, it follows from the claim that $\operatorname{cap}_\Psi^e(E;\Omega) \leq \operatorname{cap}_\Psi^t(E;\Omega)$ for all $t>0$, and so $\operatorname{cap}_\Psi^{\inf} = \operatorname{cap}_\Psi^e$.

Only the first inequality in the claim requires justification. If $t \leq e$, the claim is obviously true because $\log_+^{-\alpha} t = 1$ and $\log_+(sb) \leq \log_+(b)$, $b>0$, $0<s<1$. If $t>e$, the claim follows readily from the elementary inequality

$$\forall\, s \geq 1,\, b>0: \quad \log_+(sb) \;\leq\; \log_+(es)\log_+ b. \tag{4.4}$$

To prove (4.4), let us fix $s>1$ and $b>0$. Considering the cases $sb<e$ and $sb \geq e$ separately, we see that $\log_+(sb) \leq \log s + \log_+ b$. This last inequality implies (4.4) because $\log_+(es) = 1+\log s$ and $\log_+ b \geq 1$. $\square$

Lemma 4.5. *Let $\Psi(s) = s^p\log_+^\alpha(s)$ for some $\alpha \geq 0$, $p>1$. A set $E \subset \mathbb{R}^n$ is locally uniformly infimally Ψ-fat if and only if E is locally uniformly p-fat.*

Proof. We claim that $\operatorname{cap}_\Psi^{\inf}(F;2B) > c > 0$ if and only if $\operatorname{cap}_p(F;2B) > c' > 0$ whenever $F \subset B \equiv B(0,1)$ is compact (with c and c' dependent only on each other, α, p, and n). The lemma follows immediately from this claim.

Suppose that $\operatorname{cap}_\Psi^{\inf}(F;2B) > c > 0$. Using the notation of the last lemma, we have $\int_{2B}\Psi(|t\nabla u|)/\Psi(t) \geq c$ for all $t>0$ and all $u \in A_1(F;2B)$. In particular, if $t \leq e$, we have

$$\forall\, u \in A_1(F;2B): \qquad \int_{2B} |\nabla u|^p \log_+^\alpha(t|\nabla u|) \geq c.$$

Letting t tend to zero, we deduce that $\int_{2B}|\nabla u|^p \geq c$, and so $\operatorname{cap}_p(F;2B) > c > 0$.

Conversely, suppose that $\operatorname{cap}_p(F;2B) > c > 0$. It immediately follows that $\operatorname{cap}_\Psi^t(F;2B) > c$ for all $t \leq e$, so suppose instead that $t>e$ and let $u \in A_1(F;2B)$ be arbitrary. Since $|2B| \leq 4^n$, $\int_G |\nabla u|^p \geq c/2$, where G is the set of all $x \in 2B$ such that $|\nabla u| \geq c_0 \equiv c^{1/p}2^{-(2n+1)/p}$. A lower bound for $\operatorname{cap}_\Psi^t(F;2B)$ readily follows from the fact that there exists $C = C(c_0) < \infty$ such that $\log_+ t \leq C\log_+(st)$, for all $t>e$, $s \geq c_0$. $\square$

Proof of Theorem 0.4. Lemmas 4.3 and 4.5 imply that local uniform p-fatness is equivalent to local uniform infimal Ψ-fatness. As discussed in Section 2, Ψ lies in $G(p_1, p_2, C)$ for all $1 \le p_1 < p < p_2$ and appropriately large C. By choosing $p_1 \in (1, p)$ and $p_2 > p$ both close enough to p, the condition $p_2(n-p_1) < np_1$ is satisfied, and so the theorem follows from Theorem 2.4. □

Proof of Theorem 0.3. Suppose $-1 \le \alpha \le n-1$. Local uniform n-fatness of the complement implies (0.1) by Theorem 0.4. As we saw in Section 1, (0.1) implies (0.2). Finally, suppose that (0.2) holds. Since the integral $\int_1^\infty t^{-1} \log_+^\alpha(t)\, dt$ is infinite, it follows from Theorem 3.2 that the complement is locally uniformly Ψ-fat, and so locally uniformly n-fat by Theorem 0.5. Thus we have proved conclusion (a).

As for (b) and (c), Theorem 0.4 again says that domains with locally uniformly n-fat complement support (0.1). As for the converse direction in (b), Theorem 3.2 says that the complement is locally uniformly infimally ψ-fat, and so locally uniformly n-fat by Lemmas 4.3 and 4.5. □

We now briefly consider the relationship between the capacities associated with $\Psi(t) = t^n \log_+^\alpha(t)$ and certain associated Hausdorff contents. This allows one to give a family of associated Cantor E_α such that E_α is a null set for $\mathrm{cap}_{\Psi_\beta}$ whenever $\alpha > \beta$, but not if $\alpha < \beta$, illustrating how these capacities are pairwise quite distinct, in contrast to the associated uniform fatness conditions which, as we have seen, are all equivalent as long as $\alpha \le n-1$.

We denote by $\mathcal{H}_r^h(E)$ and $\mathcal{H}^h(E)$ the *Hausdorff content* and *Hausdorff measure*, respectively, of a set $E \subset \mathbb{R}^n$ with respect to a (continuous increasing) gauge function $h : [0,\infty) \to [0,\infty)$. For the basic theory of Hausdorff contents and Hausdorff measures, see [Mt] or [AE]. We leave to the reader the rather standard proof of the following proposition, with the hints that one direction follows by taking $\log(|x|)$ as a test function, and using the subadditivity of $\mathrm{cap}_{\Psi_\alpha}$, while the converse follows from Frostman's lemma [AE, p.6], and the Orlicz version of Hölder's inequality [RR, p.58].

Proposition 4.6. *Let* $\Psi_\alpha(t) = t^n \log_+^\alpha t$ *and* $h_\alpha(t) = \log_+^{\alpha+1-n}(1/t)$ *for some* $\alpha < n-1$, *and let* $E + t = \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) \le t\}$, $t > 0$, *where* E *is a compact subset of* $B \equiv B(0,1)$. *There exists* $C = C(n, \alpha)$ *such that* $\mathrm{cap}_{\Psi_\alpha}(E; 2B)) \le C\mathcal{H}_\infty^{h_\alpha}(E)$. *Conversely if* $\beta \in (\alpha, n-1)$, $|E + t| \le C_0 t^n / h_\alpha(t)$ *for all* $0 < t < 1$, *and* $\mathcal{H}_\infty^{h_\alpha}(E) > c_0 > 0$, *then there exists* $c = c(n, \alpha, \beta, c_0, C_0) > 0$ *such that* $\mathrm{cap}_{\Psi_\beta}(E; B(0,2)) > c$.

Defining h_γ as in Proposition 4.6, it follows as in [Mt, 4.11] that there exist Cantor sets $F_\gamma \subset [0,1]$ such that $1/4 \le \mathcal{H}^{h_\gamma}(F_\gamma) \le 1$, and such that the kth approximation to F_γ consists of 2^k intervals each of length s_k, where $h_\gamma(s_k) = 2^{-k}$. Thus $|F_\gamma + t| \lesssim 4t^n/h_\gamma(t)$, $\mathcal{H}^{h_\alpha}(F_\gamma) = 0$, and $\mathcal{H}^{h'_\beta}(F_\gamma \cap \overline{B}(x,r)) = \infty$ whenever $x \in F_\gamma$, $r > 0$, and $\beta' > \gamma$. Letting $E_\gamma = I(F_\gamma)$, where I is the usual identification of the real line with the first coordinate axis in $\mathbb{R}^n$, we see that E_γ satisfies similar Hausdorff measure and content conditions. Since $\mathcal{H}^{h_\alpha}(E_\gamma) = 0$, it follows from Proposition 4.6 that E_γ is a null set for $\mathrm{cap}_{\Psi_\alpha}$. Similarly if we choose $\beta' \in (\gamma, \beta)$, then the equality $\mathcal{H}^{h'_\beta}(E_\gamma \cap \overline{B}(x,r)) = \infty$ for all $x \in E_\gamma$, $r > 0$, together with Proposition 4.6, imply that E_γ is not a null set for $\mathrm{cap}_{\Psi_\alpha}$.

Let us conclude by listing a few questions that remain open. Unless indicated otherwise, we assume that $\Psi(t) = t^p \log_+^\alpha(t)$.

(1) In view of the fact that Ψ-fatness is a null condition when $p = n$, $\alpha > n-1$, we conjecture that (0.2) holds on every bounded domain in this case.

(2) In the case $p = n$, $\alpha < -1$, we conjecture that both (0.1) and (0.2) hold on certain bounded domains whose complement is not locally uniformly n-fat.

(3) It seems plausible that if $1 < p \le n$ and $\alpha \le p-1$, then a set is locally uniformly Ψ-fat if and only if it is locally uniform p-fat. We have seen this to be true if additionally we have either $p = n$ or $\alpha \le 0$, but the case $0 < \alpha \le p-1$, $p < n$ is open. The upper bound $p-1$ is essential, at least when p is an integer, as can be seen by the example of $(n-p)$-planes which are locally uniformly Ψ-fat when $\alpha > p-1$, but are not locally uniformly p-fat. The proof of the Ψ-fatness of $(n-p)$-planes for $\alpha > p-1$, $p \in \mathbb{N}$, is a straightforward generalization of the $p = n$ case considered in Theorem 0.5.

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Department of Mathematics, National University of Ireland, Maynooth, Co. Kildare, Ireland.

E-mail address: `sbuckley@maths.may.ie`

Department of Mathematics University of Jyväskylä, P.O. Box 35 Fin-40014, Jyväskylä Finland

E-mail address: `pkoskela@maths.jyu.fi`