Isoclinism and weak isoclinism invariants

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Abstract. We find new family invariants for group isoclinism, and also explore weak isoclinism.

1. Introduction

Isoclinism is an equivalence relation for groups that was introduced by Hall [4], and is widely used in the group theory literature. The isoclinism equivalence class containing a given group G is called the *family of* G.

There are many known *family invariants*, i.e. properties possessed by all members of a family $\mathcal{F}$ if they are possessed by any one member of $\mathcal{F}$. The isomorphism type of the commutator subgroup G' of a group G is a family invariant, but that of the center $Z(G)$ is not. Lescot [6, Lemma 2.8] showed that the *order of* $G' \cap Z(G)$ is a family invariant. Here we strengthen that result by showing that the *isomorphism type of* $G' \cap Z(G)$ is a family invariant.

Theorem 1.1. *If the groups G_1 and G_2 are isoclinic, then $G_1' \cap Z(G_1)$ and $G_2' \cap Z(G_2)$ are isomorphic.*

Let $1 \le i_{\mathrm{Ab}}(G) \le \infty$ be the minimal index of an abelian subgroup in a group G. Desmond MacHale conjectured that $i_{\mathrm{Ab}}(G)$ is a family invariant, and we show that this is indeed true.

Proposition 1.2. *If the groups G_1 and G_2 are isoclinic, then $i_{Ab}(G_1) = i_{Ab}(G_2)$.*

After some preliminaries in Section 2, we prove the above results in Section 3. In Section 4, we compare isoclinism with what we call weak isoclinism.

2. Preliminaries

Throughout the remainder of the paper, G is always a group. We use mostly standard notation: e_G is the identity of G, $H \le G$ means that H is a subgroup of G, $|G|$ is the order of G, $(G : H) = |G|/|H|$ is the index of $H \le G$, $[x, y] = x^{-1}y^{-1}xy$ is a *commutator*, G' is the *commutator subgroup of* G (generated by all commutators in G), $Z(G)$ is the *center of* G, and we call $G/Z(G)$ the *central quotient group* of G. We write $G \times H$ for the direct product of groups G, H, and $G \approx H$ indicates that G and H are isomorphic groups.

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We write C_n for the cyclic group of order n, and D_n for the dihedral group of even order $n \geq 3$. For every other explicitly mentioned group, we use the GAP ID: specifically, $\mathrm{Gp}(n, m)$ denotes the group with GAP ID (n, m). Some computations in this paper were made using GAP; for more on GAP, see [3].

For the definition of isoclinism, we use $G^{\times 2}$ as an alternative notation for the set $G \times G$; here, the group structure of $G \times G$ is irrelevant. If $\phi : G \to H$, then $\phi^{\times 2} : G^{\times 2} \to H^{\times 2}$ is the natural product map.

Since $[g_1, g_2] = [g_1 z_1, g_2 z_2]$ for all $g_1, g_2 \in G$ and all $z_1, z_2 \in Z(G)$, the commutator map induces a natural map

$$\begin{aligned} \kappa_G : (G/Z(G))^{\times 2} &\to G' \\ (g_1 Z(G), g_2 Z(G)) &\mapsto [g_1, g_2] \,. \end{aligned}$$

Definition 2.1. *Let G, H be groups. We say that (ϕ, ψ) is a weak isoclinism from G to H if $\phi : G/Z(G) \to H/Z(H)$ and $\psi : G' \to H'$ are isomorphisms. If, additionally, the diagram in Figure 1 commutes, we say that (ϕ, ψ) is an isoclinism from G to H.*

Isoclinism and weak isoclinism define equivalence relations, and we refer to the equivalence class of G under (weak) isoclinism as the *(weak) family of G*. A finite group is a *(weak) stem group* if it is of minimal order in its (weak) family.

We write $G \sim H$ if G is isoclinic to H. A readily verified example of isoclinism that we need is that $G \sim G \times A$ whenever A is an abelian group.

$$\begin{tikzcd} (G/Z(G))^{\times 2} \arrow[r, "\phi^{\times 2}", "\approx"'] \arrow[d, "\kappa_G"'] & (H/Z(H))^{\times 2} \arrow[d, "\kappa_H"] \\ G' \arrow[r, "\psi", "\approx"'] & H' \end{tikzcd}$$

Figure 1. Isoclinism

Many group properties are *(weak) family invariants*, i.e. properties possessed by all members of a (weak) family $\mathcal{F}$ if they are possessed by any one member of $\mathcal{F}$. It is immediate from the definition that the isomorphism types of G' and $G/Z(G)$ are weak family invariants. Because weak isoclinism preserves the isomorphism types of G' and $G/Z(G)$, it is also trivial that solvability and nilpotency are weak family invariants. In fact, derived length l and nilpotency class c are also weak family invariants under the restrictions $l \geq 2$ and $c \geq 2$. (Note that we need these restrictions on l and c because the trivial group is isoclinic to all other abelian groups.)

Less trivially, if $k(L)$ is the number of conjugacy classes in a group L, then $k(G)/|G| = k(H)/|H|$ whenever G and H are isoclinic finite groups; see, for instance, [6, Lemma 2.4]. However, $k(G)/|G|$ and $k(H)/|H|$ might be different if G and H are merely weakly isoclinic, as indicated by the examples in Remark 4.3 below. A list of isoclinism invariants can be found in [5, 1.1.1]. For more on group isoclinism, see for instance [1].

3. Proofs

Proof of Theorem 1.1. Let (ϕ, ψ) be an isoclinism from G_1 to G_2. For $i = 1, 2$, define the quotient map $p_i : G_i \to G_i/Z(G_i)$ and, as in Section 2, define the natural commutator map

$$\kappa_i : (G_i/Z(G_i))^{\times 2} \to G_i' .$$

Since ϕ is a homomorphism, we have

$$p_2(\kappa_2(\phi(X), \phi(Y))) = \phi(p_1(\kappa_G(X, Y))) , \qquad X, Y \in G_1/Z(G_1) .$$

It now follows from the definition of isoclinism that

$$p_2(\psi(\kappa_1(X, Y))) = \phi(p_1(\kappa_1(X, Y))) , \qquad X, Y \in G_1/Z(G_1) ,$$

or equivalently,

$$p_2(\psi([x, y])) = \phi(p_1([x, y])) , \qquad x, y \in G_1 .$$

Because p_2, p_1, ψ, and ϕ are all homomorphisms, we deduce that

$$p_2(\psi(u)) = \phi(p_1(u)) , \qquad u \in G_1' . \tag{1}$$

Using (1) and the definition of isoclinism, we see that for all $u \in G_1'$ and $W \in G_1/Z(G_1)$,

$$\begin{aligned} \kappa_2(p_2(\psi(u)), \phi(W)) &= \kappa_2(\phi(p_1(u)), \phi(W)) \\ &= \psi(\kappa_1(p_1(u), W)) . \end{aligned}$$

Since ψ is injective, we deduce that for all $u \in G_1'$ and $W \in G_1/Z(G_1)$,

$$\kappa_1(p_1(u), W) = e_1 \quad \Longleftrightarrow \quad \kappa_2(p_2(\psi(u)), \phi(W)) = e_2 , \tag{2}$$

where e_i is the identity of G_i.

The condition

$$\forall\, W \in G_1/Z(G_1) : \quad \kappa_1(p_1(u), W) = e_1$$

is easily seen to be equivalent to $u \in Z(G_1)$. Using also the surjectivity of ϕ, it similarly follows that the condition

$$\forall\, W \in G_1/Z(G_1) : \quad \kappa_2(p_2(\psi(u)), \phi(W)) = e_2$$

is equivalent to $\psi(u) \in Z(G_2)$. Bearing in mind these two equivalences, it follows from (2) that for $u \in G_1'$,

$$u \in G_1' \cap Z(G_1) \quad \Longleftrightarrow \quad \psi(u) \in G_2' \cap Z(G_2) .$$

Combining this last equivalence with the fact that $\psi : G_1' \to G_2'$ is an isomorphism, it follows that

$$\psi|_{G_1' \cap Z(G_1)} : G_1' \cap Z(G_1) \to G_2' \cap Z(G_2)$$

is also an isomorphism. □

The proof of Proposition 1.2 will follow easily from the following lemma taken from [2, 1.2]; see also [4, p. 134].

Lemma 3.1. *Suppose (ϕ, ψ) is an isoclinism from G_1 to G_2. If $Z(G_1) \le H_1 \le G_1$ and $\phi(H_1/Z(G_1)) = H_2/Z(G_2)$, then $H_1 \sim H_2$. Consequently, $\Phi(H_1) := H_2$ defines a correspondence $\Phi : \mathcal{S}(G_1) \to \mathcal{S}(G_2)$, where $\mathcal{S}(G_i)$ is the sets of subgroups of G_i containing $Z(G_i)$, $i = 1, 2$.*

Proof of Proposition 1.2. Let $\Phi : \mathcal{S}(G_1) \to \mathcal{S}(G_2)$ be as in Lemma 3.1. By the Correspondence Theorem [7, I.5.5], Φ is *index-preserving*, i.e. $(G_2 : \Phi(H)) = (G_1 : H)$ for all $H \in \mathcal{S}(G_1)$. Suppose $i_{\mathrm{Ab}}(G_1) < \infty$, and let us choose an abelian subgroup A_1 such that $(G_1 : A_1) = i_{\mathrm{Ab}}(G_1)$. Letting $A_2 := \Phi(A_1)$, we have $(G_2 : A_2) = i_{\mathrm{Ab}}(G_1)$. Moreover, $A_1 \sim A_2$ and $Z(A_1) = A_1$, so $Z(A_2) = A_2$ and A_2 is abelian. Thus $i_{\mathrm{Ab}}(G_2) \le i_{\mathrm{Ab}}(G_1)$ and, by virtue of the fact that isoclinism is a reflexive relation, it follows that $i_{\mathrm{Ab}}(G_2) = i_{\mathrm{Ab}}(G_1)$. □

Letting $\Phi : \mathcal{S}(G_1) \to \mathcal{S}(G_2)$ be as above, the Correspondence Theorem also guarantees that a subgroup $H_1 \in \mathcal{S}(G_1)$ is normal in G_1 if and only if $\Phi(H_1)$ is normal in G_2. Letting $1 \le j(G) \le \infty$ be the minimal index of an abelian normal subgroup in a group G, we therefore deduce the following variant of Proposition 1.2.

Proposition 3.2. *If the groups G_1 and G_2 are isoclinic, then $j(G_1) = j(G_2)$.*

4. Weak isoclinism

In this section, we explore the differences between weak isoclinism and the formally stronger concept of isoclinism. We first show that the two concepts are not equivalent, and that weak isoclinism does not preserve the index $i_{\mathrm{Ab}}(G)$ of Proposition 1.2. We suspect that parts (a) and (c) of the following result are known, but we do not have a reference, so we include proofs for completeness.

Theorem 4.1.

(a) There are weakly isoclinic groups G_1 and G_2 of order 64 that are not isoclinic.

(b) Furthermore we can choose G_1 and G_2 in (a) so that $i_{Ab}(G_1) = 2$ and $i_{Ab}(G_2) = 4$.

(c) If G_1 and G_2 are weakly isoclinic groups with prime power orders strictly less than 64, then they are isoclinic.

Proof. The proofs of (a) and (b) are straightforward using GAP. Let $G_1 = \mathrm{Gp}(64, 146)$ and $G_2 = \mathrm{Gp}(64, 149)$. Then $G_i/Z(G_i) \approx D_8 \times C_2$ and $G_i' \approx C_4 \times C_2$ for $i = 1, 2$, so these two groups are weakly isoclinic. However $i_{\mathrm{Ab}}(G_1) = 2$ and $i_{\mathrm{Ab}}(G_2) = 4$. By Proposition 1.2, G_1 and G_2 cannot be isoclinic.

As for (c), it is clear that groups of prime power order for different primes cannot be weakly isoclinic unless they are abelian, in which case they are certainly isoclinic. Thus it suffices to consider nonabelian groups of prime power order for each prime p separately. There are two nonabelian groups of order 27, and they are known to be isoclinic. All other nonabelian groups of prime power order less than 64 are 2-groups.

As mentioned in [4, p. 136], there are eight families containing groups of order 2^i for $i \leq 5$. Each has a representative of order 32 (just take a direct product of a stem group in the family with a suitable cyclic group). A GAP computation shows that groups of order 32 have eight different isomorphism types of central quotient groups, so there are at least eight weak families containing such groups. Thus the number of weak families and the number of families containing such groups must coincide, and a weakly isoclinic pair of groups of order 32 must also be isoclinic. □

Remark 4.2. It may well be that Theorem 4.1(c) could be strengthened to state that weakly isoclinic groups of order less than 64 are necessarily isoclinic. However, isoclinism has been used mainly for groups of prime power order, and there appears to be little written on the isoclinism classes for non-prime power orders. Consequently, it would require a fair deal of calculation to investigate such a stronger result.

Remark 4.3. We can prove Theorem 4.1(a) using other pairs of groups of order 64. For instance, we could take $G_1 = D_8 \times D_8$ and $G_2 = \mathrm{Gp}(64, 215)$. As is well known, $D_8/Z(D_8) \approx C_2 \times C_2$, $D_8' \approx C_2$, and D_8 has five conjugacy classes. It follows that $G_1/Z(G_1) \approx (C_2)^4$ and $G_1' \approx C_2 \times C_2$, and that G_1 has 25 conjugacy classes. GAP reveals that G_2 is weakly isoclinic to G_1, but has only 22 conjugacy classes. Since number of conjugacy classes divided by group order is a family invariant, it follows that G_1 and G_2 are not isoclinic. However $i_{\mathrm{Ab}}(G) = 4$ for every group G of order 2^j, $j \leq 6$, with $G/Z(G) \approx (C_2)^4$ have (as can be verified with GAP), so these groups cannot be used to show that the weak isoclinism analogue of Proposition 1.2 fails.

We now show that the weak isoclinism analogue of Theorem 1.1 is false.

Theorem 4.4. *There exist weakly isoclinic groups G_1 and G_2 of order 256 such that $G_1' \cap Z(G_1)$ and $G_2' \cap Z(G_2)$ are not isomorphic. There are no weakly isoclinic groups of strictly smaller prime power orders with the same property.*

Proof. GAP gives many pairs of groups with the desired properties. For instance, we can take $G_1 := \mathrm{Gp}(256, 3000)$ and $G_2 := \mathrm{Gp}(256, 3300)$. Then

$G_i/Z(G_i) \approx \mathrm{Gp}(32,34)$ and $G_i' \approx C_4 \times C_4 \times C_2$ for $i = 1,2$. However, $G_1' \cap Z(G_1) \approx C_4 \times C_2$, while $G_2' \cap Z(G_2) \approx C_2 \times C_2 \times C_2$.

In view of Theorem 4.1, proving minimality requires only that we verify that the phenomenon does not occur among groups of orders 64, 81, 128, 243, and 125, and GAP confirms this. □

Remark 4.5. Let us pause to explain how to use groups of order 2^n for $n < 8$ to construct non-minimal examples of weakly isoclinic groups G_1, G_2 such that $G_1' \cap Z(G_1)$ and $G_2' \cap Z(G_2)$ are non-isomorphic. The advantage of doing this is that examining groups of order 2^n for $n < 8$ involves considerably less effort than examining the 56 092 groups of order $256 = 2^8$.

GAP gives us the following twelve isomorphisms involving the groups $G_i := K_i \times L_i$ for $i = 1,2$, where $K_1 := \mathrm{Gp}(64,34)$, $L_1 := \mathrm{Gp}(128,437)$, $K_2 := \mathrm{Gp}(64,32)$, and $L_2 := \mathrm{Gp}(128,742)$:

$$\begin{aligned}
K_1/Z(K_1) &\approx \mathrm{Gp}(32,6)\,, & K_1' &\approx C_2 \times C_2 \times C_2\,, & K_1' \cap Z(K_1) &\approx C_2\,,\\
L_1/Z(L_1) &\approx \mathrm{Gp}(32,34)\,, & L_1' &\approx C_4 \times C_4\,, & L_1' \cap Z(L_1) &\approx C_2 \times C_2\,,\\
K_2/Z(K_2) &\approx \mathrm{Gp}(32,6)\,, & K_2' &\approx C_4 \times C_2\,, & K_2' \cap Z(K_2) &\approx C_2\,,\\
L_2/Z(L_2) &\approx \mathrm{Gp}(32,34)\,, & L_2' &\approx C_4 \times C_2 \times C_2\,, & L_2' \cap Z(L_2) &\approx C_4\,.
\end{aligned}$$

It is clear from the first and second columns above that G_1 and G_2 are weakly isoclinic. However, the third column reveals that

$$\begin{aligned}
G_1' \cap Z(G_1) &\approx C_2 \times C_2 \times C_2\,,\\
G_2' \cap Z(G_2) &\approx C_4 \times C_2\,,
\end{aligned}$$

so we are done.

Although the conclusion of Theorem 1.1 fails when we replace isoclinism by weak isoclinism, we recover a true statement when we also weaken the conclusion:

Proposition 4.6. *If the groups G_1 and G_2 are weakly isoclinic, and $|G_1'| < \infty$, then $|G_1' \cap Z(G_1)| = |G_2' \cap Z(G_2)|$.*

Proof. Weak isoclinism implies that $(G_1/Z(G_1))' \approx (G_2/Z(G_2))'$. But by the second isomorphism theorem,

$$(G_1/Z(G_1))' = (G_1'Z(G_1))/Z(G_1) \approx G_1'/(G_1' \cap Z(G_1))\,,$$

and so $G_1'/(G_1' \cap Z(G_1)) \approx G_2'/(G_2' \cap Z(G_2))$. Weak isoclinism also implies that $|G_1'| = |G_2'|$, so the result follows. □

Remark 4.7. Coding most of the GAP-dependent statements in the proofs of Theorems 4.1 and 4.4 was a triviality. The only exception was the computation of $i_{\mathrm{Ab}}(G)$, and the code used for that involved only a few lines. Specifically, we first defined the index function:

```
iAb := function(G)
  local Cl, AbOrd;
  Cl := ConjugacyClassesSubgroups(G);
  AbOrd := Filtered(Cl, x -> IsAbelian(Representative(x)));
  Apply(AbOrd, x -> Order(Representative(x)));
  return Order(G)/Maximum(AbOrd);
end;;
```

The index $i_{\mathrm{Ab}}(G_1)$, for instance, could then be computed by the function call

```
iAb(SmallGroup(64,146));
```

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