POINTWISE MULTIPLIERS FOR REVERSE HÖLDER SPACES II

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Abstract
We classify weights which map strong reverse Hölder weight classes to weak reverse Hölder weight spaces under pointwise multiplication.

1. Introduction

In this paper, we classify those weights f for which fw satisfies a weak reverse Hölder condition for every w satisfying some strong reverse Hölder condition (see Theorem 1.2). This "weak-strong" problem and the corresponding "weak-weak" and "strong-strong" problems were investigated in [1], where a simple necessary and sufficient condition on the weight f was found in each of the latter two cases, but the first problem was only partially answered. This paper rectifies the matter by giving a simple necessary and sufficient condition for f to satisfy a mixed condition of this type.

We first introduce some terminology and notation. Throughout the paper, Ω is a fixed open subset of $\mathbf{R}^n$. By a weight on Ω, we mean any non-negative measurable function defined on Ω, which is not identically zero. Since we are only concerned with integrals of weights throughout, sets are always assumed to be measurable, and sets of measure zero do not concern us. A *cube* Q is always assumed to have faces perpendicular to coordinate axes, and its sidelength will be denoted by $l(Q)$. If $t > 0$, tQ is the cube concentric with Q such that $l(tQ) = t \cdot l(Q)$. We say that two cubes are *adjacent* if their closures intersect but their interiors are disjoint. For any set E and weight w, we write $|E|$ for the Lebesgue measure of E, $w(E) = \int_E w$, and

$$\|w\|_{p,E} = \left(\frac{1}{|E|} \int_E w^p(x)\,dx \right)^{1/p}, \qquad p \neq 0$$

As usual, $\|w\|_{\infty,E} = \operatorname*{ess\,sup}_{x\in E} w(x)$. Thus $\|w\|_{p,E}$ is a monotonically increasing function of p. If $\sigma > 1$ and $\sigma Q \subseteq \Omega$, we say that Q is "σ-dilatable". We denote the Hardy-Littlewood maximal operator by M and, for any $1 < p < \infty$, we shall write $p' = p/(p-1)$.

We shall be concerned with weights $w \in L^p_{\mathrm{loc}}(\Omega)$ for which

$$\exists K \in \mathbf{R} : \|w\|_{p,Q} \le K \|w\|_{q,\sigma Q}, \qquad \text{for all } \sigma'\text{-dilatable } Q \tag{1.1}$$

for some $0 < q < p$, $1 \le \sigma \le \sigma'$. Weights satisfying such conditions have been studied by many authors; some important advances are to be found in [6], [4], [2], [9], and

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[5]. For a more thorough discussion of such weights, and of the statements made in the following paragraphs, we refer the reader to [1].

Assuming that $0 < q < p$, the class of weights satisfying (1.1) is denoted WRH_p^Ω if $1 < \sigma \le \sigma'$, $RH_p^{\Omega,\mathrm{loc}}$ if $1 = \sigma < \sigma'$, and RH_p^Ω if $1 = \sigma = \sigma'$. These classes are independent of q, σ or σ', as long as those parameters satisfy the defining equalities and inequalities. In the first two cases, we say that w satisfies a *strong reverse Hölder condition* of order p on Ω, while in the last case we say that w satisfies a *weak reverse Hölder condition*. For $q = p/2$, $\sigma = 2$, $\sigma' = 4$, the smallest constant K for which (1.1) is true will be denoted $WRH_p^\Omega(w)$, and will be referred to as the "WRH_p^Ω norm" of w. Similarly we define "norms" $RH_p^{\Omega,\mathrm{loc}}(w)$ and $RH_p^\Omega(w)$ by choosing $(q, \sigma, \sigma') = (p/2, 1, 4)$ and $(q, \sigma, \sigma') = (p/2, 1, 1)$ respectively. The values of q, σ, and σ' used have no significance — if they are changed, the new norms are equivalent to the old ones up to a constant dependent only on these parameters and the dimension (of course, the choices $\sigma = 1$ and $\sigma' = 1$ in the last two definitions cannot be varied).

WRH_p^Ω, $RH_p^{\Omega,\mathrm{loc}}$ and RH_p^Ω share some properties dependent only on p, so we temporarily denote any one of these classes as S_p. Obviously, $S_p \subseteq S_q$ if $0 < p < q$, and it is easy to produce examples to show that this containment is strict. Nevertheless, it is also true (see [2], [4] and [9]) that $S_p = \bigcup_{q>p} S_q$. In fact, if $w \in S_p$, then $w \in S_{p+\epsilon}$ for some $\epsilon > 0$ dependent only on n, p, and the S_p-norm of w; we can even choose ϵ so small that $S_{p+\epsilon}(w) \le 2S_p(w)$.

Strong reverse Hölder conditions are related to the A_p condition of Muckenhoupt (see [2]). It follows that if w satisfies (1.1) for $1 = \sigma \le \sigma'$, and some $0 < q < p$, then w actually satisfies (1.1) for all $q \ge -\epsilon$ (and σ, σ' unchanged). As before, the size of ϵ depends only upon n, p and the norm of w in its weight class. This "improvement" is not possible for weak reverse Hölder conditions — if w satisfies (1.1) for $q < 0 < p$ and $1 < \sigma < \sigma'$, it also satisfies (1.1) for $\sigma = 1$, with the other parameters unchanged.

We now state the main theorem of this paper.

Theorem 1.2. *Let S_p be either RH_p^Ω or $RH_p^{\Omega,\mathrm{loc}}$. Suppose also that f is a weight and $0 < p, q \le \infty$. If $f \cdot S_p \subseteq WRH_q^\Omega$, then $q \le p$. Furthermore,*

(i) *if $0 < q \le p < \infty$, then $f \cdot S_p \subset WRH_q^\Omega$ if and only if $f \in \bigcap_{r<s} WRH_r^\Omega$, where $s = pq/(p-q)$ ($s = \infty$ if $p = q$);*

(ii) *if $0 < q \le \infty$, then $f \cdot S_\infty \subset WRH_q^\Omega$ if and only if $f \in WRH_q^\Omega$.*

In either case, if $w \in S_p$, then $WRH_q^\Omega(fw)$ is dependent only upon n, p, q, and the norms of f and w in their respective weight classes.

Note that $f \cdot S_p \subseteq WRH_q^\Omega$ implies that $f \cdot 1 \in WRH_q^\Omega$. This gives the "only if" direction of the theorem when $p = \infty$, but not when $p < \infty$. In [1], an iteration argument gave more information for the corresponding unmixed problems (the next step was to consider $f \cdot f^{q/p}$); in fact, this iteration alone sufficed for the strong-strong problem. We cannot, of course, employ such a method here. As we shall see in section 3, bridging the gap between the index q and all indices less than s is what requires most of our effort.

2. A pair of lemmas

We first need some notation: if $R > 0$, $\sigma > 1$, then for any weight w and cube Q, $E(R, Q) \equiv E(R, Q; w, \sigma)$ denotes the set $\{x \in Q \mid w(x) \geq R\|w\|_{1,\sigma Q}\}$.

Our first lemma gives several conditions equivalent to WRH_p^Ω. This lemma will have a familiar feel to it for those conversant with the literature on reverse Hölder classes. For example, the RH_p^Ω analogue of the equivalence of (i) and (ii) was proven by Coifman and C. Fefferman in [2] where inequalities related to the RH_p^Ω version of (iii) are also examined (see also [3]). The equivalence of (i) and (ii) for WRH_p^Ω is due to Sawyer ([7]).

Note that in this lemma, the constants C, ϵ and t in (ii)–(iv) are determined by n, p, σ, σ', and $WRH_p^\Omega(w)$ alone. Also, analogues of the lemma for $RH_p^{\Omega,\mathrm{loc}}$ and RH_p^Ω can be proved in almost exactly the same manner.

Lemma 2.1. *Suppose that w is a weight, $1 < \sigma \leq \sigma'$, and $1 < p$. Then the following are equivalent.*

(i) $w \in WRH_p^\Omega$;

(ii) *there exist $\epsilon > 1/p'$, $C > 0$ such that* $\dfrac{w(E)}{w(\sigma Q)} \leq C\left(\dfrac{|E|}{|Q|}\right)^\epsilon$, *for all subsets E of σ'-dilatable cubes Q;*

(iii) *there exist $t > p-1$, $C > 0$ such that* $\dfrac{w(E(R,Q))}{w(\sigma Q)} \leq CR^{-t}$, *for all σ'-dilatable cubes Q and all $R > 1$;*

(iv) *there exist $t > p$, $C > 0$ such that* $\dfrac{|E(R,Q)|}{|Q|} \leq CR^{-t}$, *for all σ'-dilatable cubes Q and all $R > 1$.*

Proof. Suppose that $w \in WRH_p^\Omega$ and so $w \in WRH_s^\Omega$ for some $s > p$. If E is a subset of a σ'-dilatable cube Q, then

$$\frac{w(E)}{|Q|} = \|w\chi_E\|_{1,Q} \leq \|w\|_{s,Q}\|\chi_E\|_{s',Q} \leq K\|w\|_{1,\sigma Q}\left(\frac{|E|}{|Q|}\right)^{1/s'}$$

and so

$$\frac{w(E)}{w(\sigma Q)} \leq C\left(\frac{|E|}{|Q|}\right)^{1/s'}$$

Thus (i) implies (ii) with $\epsilon = 1/s' > 1/p'$.

Next we show that (ii) implies (iii). Fixing a σ'-dilatable cube Q, we normalise so that $w(\sigma Q) = |Q| = 1$. Upper bounds for $w(E(R,Q))$ can be improved using (ii) as follows:

$$w(E(R,Q)) \leq A \Longrightarrow |E(R,Q)| \leq A/R \Longrightarrow w(E(R,Q)) \leq C(A/R)^\epsilon. \tag{2.2}$$

Starting with the trivial estimate $w(E(R,Q)) \le 1$, and iterating (2.2), we get that $w(E(R,Q)) \le (C/R^\epsilon)^{s_k}$, where $s_k = \sum_{j=0}^{k-1} \epsilon^j$. Letting $k \to \infty$, we see that $w(E(R,Q)) \le C^{1/(1-\epsilon)}/R^t$, where $t = \epsilon/(1-\epsilon) > p-1$ since $\epsilon > 1/p'$.

(iv) follows immediately from (iii) using the first implication in (2.2), so let us finish by showing that (iv) implies (i). We fix a σ'-dilatable cube Q and normalise so that $\|w\|_{1,\sigma Q} = 1$. Letting $E_k = \{x \in Q \mid 2^{k-1} < w(x) \le 2^k\}$ for $k > 0$, it follows from (iv) that $|E_k|/|Q| \le C2^{-(k-1)t}$ and so

$$\frac{1}{|Q|}\int_Q w^p \le 1 + \sum_{k=1}^{\infty} 2^{kp}\frac{|E_k|}{|Q|} \le 1 + C' \sum_{k=1}^{\infty} 2^{-k(t-p)} \le C''. \qquad \square$$

It is known ([3, theorem IV.2.16]) that, if f is any locally integrable function such that Mf is finite a.e. and $0 < \alpha < 1$, then $(Mf)^\alpha$ is an A_1 weight (and all A_1 weights are essentially of this type). It follows that if $p > 0$ then $(Mf)^\alpha \in RH_p^{\mathbf{R}^n}$ for all $0 < \alpha < 1/p$; also, $RH_p^{\mathbf{R}^n}((Mf)^\alpha)$ is bounded by a constant dependent only on α, p and n. The following useful technical lemma shows that, in certain circumstances, we can patch together a sequence of such weights to produce another $RH_p^{\mathbf{R}^n}$ weight. In this lemma and the following discussion, we write $A \lesssim B$ if A and B are two quantities for which $A \le CB$, where C is some constant independent of k, and the choices of Q_k and E_k. Also, $A \approx B$ means that $A \lesssim B \lesssim A$.

Lemma 2.3. *Suppose that $\sigma > 1$, $p, \alpha > 0$ and $\alpha < 1/p$. Suppose also that E_k is a subset of a cube Q_k and has non-zero measure, for all integers k in some interval I. Then in either of the following two (mutually exclusive) cases, there exists a weight $w \in RH_p^{\mathbf{R}^n}$ and constants c_k such that $w(x) \approx c_k(M\chi_{E_k}(x))^\alpha$ for $x \in Q_k \backslash \sigma Q_{k+1}$ and all $k \in I$:*

(i) *the cubes σQ_k are pairwise disjoint,*
(ii) *$\sigma Q_{k+1} \subset Q_k \backslash E_k$.*

Proof. We first prove (i). In this case, $Q_k \backslash \sigma Q_{k+1} = Q_k$, so the ordering of the cubes is irrelevant. Let us choose $\sigma_1 \in (1,\sigma)$ and write $U = \bigcup_{k\in I} \sigma Q_k$, $V = \bigcup_{k\in I} \sigma_1 Q_k$. We define

$$w(x) = \begin{cases} c_k(M\chi_{E_k})^\alpha, & x \in \sigma Q_k \\ 1, & x \notin U, \end{cases}$$

choosing the constants c_k so that $w(x) \approx 1$ on the annular regions $\sigma Q_k \backslash \sigma_1 Q_k$ (note that $M\chi_{E_k}$ is essentially constant on this annulus).

We need to show that $\|w\|_{p,Q} \lesssim \|w\|_{p/2,Q}$ for all cubes Q. This is obvious if $Q \subset U$, since then $Q \subset \sigma Q_k$ for some k, and $w \in RH_p^{\sigma Q_k}$ by construction. We now show that $\|w\|_{p,Q} \lesssim 1 \lesssim \|w\|_{p/2,Q}$ for all $Q \not\subset U$. First note that if Q intersects

$\sigma_1 Q_k$, then $Q \cap (\sigma Q_k \backslash \sigma_1 Q_k)$ includes a cube P_k of sidelength comparable to that of Q_k. Since $w(x) \approx 1$ for $x \in P_k$,

$$\int_{Q \cap \sigma Q_k} w^{p/2} \geq \int_{P_k} w^{p/2} \approx |P_k| \approx |Q \cap \sigma Q_k|. \tag{2.4}$$

Since $w(x) \approx 1$ for $x \in Q \backslash V$, it follows easily from (2.4) that $1 \lesssim \|w\|_{p/2,Q}$. The second inequality follows in a similar fashion:

$$\int_{Q \cap \sigma Q_k} w^p \leq \int_{\sigma Q_k} w^p \approx \int_{P_k} w^p \approx |P_k| \approx |Q \cap \sigma Q_k|,$$

since $w \in RH_p^{\sigma Q_k}$ and so $w^p\,dx$ is a doubling measure on σQ_k (see [2]).

Let us now prove (ii) for $I = \mathbf{R}$ (and so $\bigcup_{k \in \mathbf{Z}} Q_k = \mathbf{R}^n$). We first choose σ_1, σ_2, σ_3 such that $1 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma$. For any constant $c_k > 0$, we note that $w_k(x) = c_k (M\chi_{E_k}(x))^\alpha \in RH_p^{\mathbf{R}^n}$, and that w_k is essentially constant on $\sigma_1 Q_{k+1}$ (since σQ_{k+1} and E_k are disjoint) and on $\sigma_1 Q_k \backslash \sigma_3 Q_k$ (since $E_k \subseteq Q_k$). We choose $c_0 = 1$, say, and then define c_k inductively for positive and negative k, in such a way that $w_k(x) \approx r w_{k-1}(y)$ for all $x, y \in \sigma_1 Q_k \backslash \sigma_3 Q_k$, where $r \in (0,1)$ is to be specified shortly. We now define $w(x) = w_k(x)$ for all $x \in \sigma_2 Q_k \backslash \sigma_2 Q_{k+1}$, and all k.

We need to show that $\|w\|_{p,Q} \lesssim \|w\|_{p/2,Q}$ for all cubes Q. We first show that there exists $C > 0$ (independent of r) such that

$$\forall\, j \in \mathbf{Z}: \qquad \int_{\sigma_2 Q_j \backslash \sigma_2 Q_{j+1}} w^p \leq C r^p \int_{\sigma_2 Q_{j-1} \backslash \sigma_2 Q_j} w^p. \tag{2.5}$$

To see this, we choose cubes $\{R_k\}_{k=-\infty}^{\infty}$ such that $R_k \subset \sigma_2 Q_k \backslash \sigma_3 Q_k$ and $l(R_k) \approx l(Q_k)$. Since $w_j, w_{j-1} \in RH_p^{\mathbf{R}^n}$, we get

$$\begin{aligned}
\int_{\sigma_2 Q_j \backslash \sigma_2 Q_{j+1}} w^p &\leq \int_{\sigma_2 Q_j} w_j^p \approx \int_{R_j} w_j^p \\
&\approx r^p \int_{R_j} w_{j-1}^p \leq r^p \int_{\sigma_2 Q_{j-1}} w_{j-1}^p \\
&\approx r^p \int_{R_{j-1}} w_{j-1}^p \leq r^p \int_{\sigma_2 Q_{j-1} \backslash \sigma_2 Q_j} w^p,
\end{aligned}$$

as required. We now fix $r \in (0,1)$ so small that $Cr^p < 1/2$ in (2.5).

Given any cube Q, there exists $k \in \mathbf{Z}$ for which $Q \subset \sigma_2 Q_{k-1}$ but $Q \not\subset \sigma_2 Q_k$. If $Q \subset \sigma_2 Q_{k-1} \backslash \sigma_3 Q_k$, the desired Hölder inequality is immediate since then $w(x) \approx w_{k-1}(x)$ for $x \in Q$. Otherwise we note that $Q \cap (\sigma_2 Q_k \backslash \sigma_3 Q_k)$ includes a cube P_k of sidelength comparable to that of Q_k. Thus,

$$\int_{Q \cap \sigma_2 Q_k} w^{p/2} \geq \int_{P_k} w^{p/2} \approx \int_{P_k} w_{k-1}^{p/2} \approx \int_{\sigma_2 Q_k} w_{k-1}^{p/2} \geq \int_{Q \cap \sigma_2 Q_k} w_{k-1}^{p/2}, \tag{2.6}$$

and

$$\int_{\sigma_2 Q_k \setminus \sigma_2 Q_{k+1}} w^p \le \int_{\sigma_2 Q_k} w_k^p \approx \int_{P_k} w_k^p \approx \int_{P_k} w_{k-1}^p \tag{2.7}$$

It follows from (2.5), (2.7), and our choice of r that

$$\int_{Q\cap\sigma_2 Q_k} w^p \lesssim \left(\sum_{j=0}^{\infty} 2^{-j}\right) \int_{P_k} w_{k-1}^p \le 2 \int_{Q\cap\sigma_2 Q_k} w_{k-1}^p. \tag{2.8}$$

Finally (2.6) and (2.8) imply

$$\|w\|_{p,Q} \lesssim \|w_{k-1}\|_{p,Q} \lesssim \|w_{k-1}\|_{p/2,Q} \lesssim \|w\|_{p/2,Q},$$

which finishes the proof of (ii) (for $I = \mathbf{R}$). The case $I \neq \mathbf{R}$ can be handled by a few easy modifications to the above proof, so we omit the details. □

Let us pause to show the importance of the assumption $\sigma > 1$ in Lemma 2.3. If we instead assume that $\sigma = 1$, leaving the other assumptions unchanged, the lemma is false in either of the two cases. In the following counterexamples, "w is approximately constant on a set $S_k \subset \mathbf{R}$" means that $w(x) \lesssim w(y)$ for almost all $x, y \in S_k$.

For the case of pairwise disjoint cubes, let $n = 1$, $Q_k = (k, k+1)$, and $E_k = (k, k+2^{-k})$, for all $k \ge 1$. Suppose that a weight w with the required properties exists. Then w is approximately constant on $(k-1/2, k)$ and on $(k, k+2^{-k})$. Also $w^p\,dx$ is a doubling measure and so w is approximately constant on $A_k = (k-1/2, k+2^{-k})$, for all $k \ge 1$. Since A_k and A_{k+1} are approximately the same length, and each is contained in the 5-dilate of the other, we can use doubling again to get that $w(A_k) \approx w(A_{k+1})$, and so w is approximately constant on $A_k \cup A_{k+1}$. This contradicts the fact that, for large k, the approximate value of w on $(k, k+2^{-k})$ must be much larger than its approximate value on $(k+1/2, k+1)$, since the same is true of $M\chi_{E_k}$.

In the nested cubes case, let $n = 1$, $Q_k = (-2^{-k}, 2^{-k})$, and $E_k = (2^{-k-1}, 2^{-k-1} + 4^{-k-1})$, and suppose that such a weight w exists. By considering $M\chi_{E_{k-1}}$ and $M\chi_{E_k}$ respectively, we see that w is approximately constant on E_{k-1} and on $(3\cdot 2^{-k-2}, 2^{-k})$. Hence, by doubling, it is approximately constant on $A_k = (3\cdot 2^{-k-2}, 2^{-k} + 4^{-k})$. Again by doubling, w is approximately constant on $A_k \cup A_{k+1}$, which contradicts the behaviour of $M\chi_{E_k}$ on $(2^{-k-1}, 2^{-k}) \subset Q_k \setminus Q_{k+1}$.

3. Proof of Theorem 1.2

In view of the last statement in Theorem 1.2, we shall use C throughout this proof to refer to any constant dependent only upon n, p, q, and the norms of the relevant weights in the weight classes from which they are chosen (C may also depend on additional parameters such as t, ϵ which, in turn, depend only on the aforementioned parameters).

We first show that the inclusion $f \cdot S_p \subset WRH_q^{\Omega}$ leads to a contradiction if $p < q$. It suffices to assume $S_p = RH_p^{\Omega}$. Let x_0 be a point in the Lebesgue set of f^q such that $f^q(x_0) > 0$. If $A_k = \{x \in \mathbf{R}^n \mid 2^{-k-1} < |x| \le 2^{-k}\}$, the family of annuli $\mathcal{F} = \{A_k\}_{k=1}^\infty$ is *regular* in the sense of Stein [8, section I.1.8], and so there exists $k_0 > 0$ such that

$$\forall\, k \ge k_0 : \qquad \frac{1}{|A_k|} \int_{A_k} f^q(x_0 + y)\, dy > f^q(x_0)/2.$$

We may also assume k_0 to be large enough that $\{x : |x - x_0| < 2^{-k_0}\sigma'\} \subset \Omega$. Choosing α so that $n/q < \alpha < n/p$, we have $w(x) \equiv |x - x_0|^{-\alpha} \in RH_p^{\mathbf{R}^n}$. However,

$$\int_{\{|x-x_0|\le 2^{-k_0}\}} (fw)^q = \sum_{k=k_0}^\infty \int_{A_k + x_0} (fw)^q \ge C \left(\sum_{k=k_0}^\infty 2^{(\alpha q - n)k} \right) f^q(x_0) = \infty$$

and so $fw \notin WRH_q^\Omega$.

The proofs for part (ii) and the "if" direction in (i) were stated in [1], and their proofs are essentially the same as the corresponding proofs for the "strong-strong" case which are given there. Since their proofs are short, we include them here for completeness.

We first consider (ii). If $f \cdot S_\infty \subset WRH_q^\Omega$ then $f \cdot 1 \in WRH_q^\Omega$. Conversely, if $f \in WRH_q^\Omega$, $w \in RH_\infty^\Omega \subseteq S_\infty$, and Q is σ'-dilatable, then

$$\begin{aligned} \|fw\|_{q,Q} \le \|f\|_{q,Q} \|w\|_{\infty,Q} &\le C\|f\|_{\epsilon/2,\sigma Q} \|w\|_{\infty,Q} \\ &\le C\|fw\|_{\epsilon,\sigma Q} \|w\|_{-\epsilon,\sigma Q}^{-1} \|w\|_{\infty,Q} \le C\|fw\|_{\epsilon,\sigma Q}, \end{aligned}$$

as long as $\epsilon > 0$ is sufficiently small. Thus $fw \in WRH_q^\Omega$.

As for the "if" part of (i), we first consider the case $q < p$. Fix $1 < \sigma < \sigma'$ and suppose that $f \in \bigcap_{r<s} WRH_r^\Omega$, $w \in RH_p^{\Omega,\mathrm{loc}}$. Thus $w \in RH_{tq}^{\Omega,\mathrm{loc}}$ for some $t > p/q$. Choose $0 < \epsilon < q$ small enough that $\|w\|_{tq,Q} \le C\|w\|_{-\epsilon,Q}$ for all σ'-dilatable Q. Since $t > p/q$, it follows that $t'q < s$, and so using Hölder and reverse Hölder inequalities we get

$$\begin{aligned} \|fw\|_{q,Q} \le \|w\|_{tq,Q} \|f\|_{t'q,Q} &\le C\|w\|_{-\epsilon,Q} \|f\|_{\epsilon/2,\sigma Q} \\ &\le C\|w\|_{-\epsilon,Q} \|fw\|_{\epsilon,\sigma Q} \|w\|_{-\epsilon,\sigma Q}^{-1} \\ &\le C\|fw\|_{q,\sigma Q}. \end{aligned}$$

The case $p = q$ now follows easily: if $w \in RH_p^{\Omega,\mathrm{loc}}$, then $w \in RH_t^{\Omega,\mathrm{loc}}$ for some $t > q$, and $f \cdot w \in WRH_p^\Omega$ since $f \in WRH_{tp/(t-p)}^\Omega$.

It remains only to prove the main part of the theorem: the "only if" direction of (i). Here, we can assume that $q < p$, since this case immediately implies the case $q = p$. Furthermore, if the result is true for a particular choice of parameters (p, q), it is also true for the parameters $(p/t, q/t)$ for any $t > 0$. To see this, note that

$f \cdot RH_p^\Omega \subset WRH_q^\Omega$ if and only if $f^t \cdot RH_{p/t}^\Omega \subset WRH_{q/t}^\Omega$, and $f \in \bigcap_{r<s} WRH_r^\Omega$ if and only if $f^t \in \bigcap_{r<u} WRH_r^\Omega$, where $u = (p/t)(q/t)/(p/t - q/t)$. This observation enables us to assume without loss of generality that $1 < p' < q < p$ (and so $s > 1$). Note also that we may assume $f = f \cdot 1 \in WRH_q^\Omega \subset WRH_{p'}^\Omega$.

Let us pause to motivate and outline the rest of the proof. First note that Lemma 2.1 says roughly that f is a weak reverse Hölder weight if and only if the subset of a cube Q where f is very big is uniformly controlled by the average size of f on $2Q$. If f is not in the required reverse Hölder class, this fact is equivalent to the existence of a sequence (referred to as a "T-sequence" below, as it is given by a sequence of triples) of cubes Q_k and subsets E_k on which the type of control of Lemma 2.1(iv) is only true with constants $C = C_k \to \infty$ $(k \to \infty)$. Our first step is then to prove the result in the "Special Case" where we assume $RH_p^\Omega(fw)$ is bounded by a constant dependent on $WRH_q^\Omega(w)$, but otherwise independent of w. An appropriate positive power of $M\chi_{E_k}$ gives a sequence of weights w_k such that $RH_p^\Omega(w_k)$ is uniformly bounded. w_k has the desirable property on Q_k of being big where f is big, and small where f is small. Consequently, we shall see that $WRH_p^\Omega(fw_k)$ is an unbounded sequence of numbers, finishing this case.

Although the Special Case is completely contained in the later cases, proceeding in this manner aids clarity since we shall be able to reduce most of the subsequent cases to situations where a similar argument will clearly work. In order to eliminate the control assumption and prove the full-strength result, the obvious plan is to patch together the weights w_k so as to create a single weight $w \in RH_p^\Omega$ for which fw cannot be in WRH_q^Ω (because examination of its values on Q_k gives a sequence of lower bounds for $WRH_q^\Omega(fw)$ which tend to infinity as k does). This plan has a snag: the cubes and subsets may have arbitrary sizes and overlaps to begin with, and so the weights w_k might not be suitable for being patched together. Consequently, we split the argument into various cases, in each of which we make successive changes to our T-sequence to create new T-sequences with more desirable properties until eventually we can patch the associated weights together, or arrive at a contradiction through other means.

Suppose, for the purposes of contradiction, that $f \notin WRH_{r_1}^\Omega$ for some $1 < r_1 < s$. Let us fix $r_2 \in (r_1, s)$ and suppose that $1 < \sigma < \sigma'$. We define a *T-sequence* for f (with parameters σ, σ') to be any sequence of triples $\{(Q_k, E_k, R_k)\}_{k=1}^\infty$ where Q_k is a σ'-dilatable cube, $E_k \subseteq Q_k$,

$$\forall k: \quad f(x) \ge R_k \|f\|_{1,\sigma Q_k} \quad \text{on } E_k \tag{3.1}$$

and

$$\forall k: \quad |E_k|/|Q_k| > k R_k^{-r_2}. \tag{3.2}$$

Since $f \notin WRH_{r_1}^\Omega$, Lemma 2.1 implies the existence of a T-sequence (for any $1 < \sigma < \sigma'$): we simply choose $E_k = E(R_k, Q_k; f, \sigma)$ for an appropriate sequence of counterexamples to 2.1(iv). It immediately follows from (3.2) that $R_k > 1$, and that $R_k \to \infty$ $(k \to \infty)$. Also note that any subsequence of a T-sequence is also a T-sequence (a "*T-subsequence*").

We shall often replace E_k by E'_k, some other subset of Q_k for which (3.1) is valid and which has some additional desirable property P. When we make such a change, there will always be some $\epsilon > 0$ for which $|E'_k|/|E_k| \geq \epsilon > 0$, for all k. Thus, if $N > 1/\epsilon$, then $\{(Q_{Nk}, E'_{Nk}, R_{Nk})\}_{k=1}^{\infty}$ is a T-sequence with property P, assuming that P is preserved by the taking of a subsequence. In such a case, we may therefore assume that the original T-sequence we chose had this extra property (thus avoiding the creation of many new names below for derived T-sequences). Similarly, we may replace Q_k by $Q'_k \supset Q_k$ if the following conditions are true:

(i) $|Q_k|/|Q'_k| \geq \epsilon$ for some $\epsilon > 0$,
(ii) (3.1) remains valid if we replace (Q_k, R_k, σ) by $(Q'_k, \delta R_k, \nu)$, for some $\delta > 0$, $\nu > 1$.

We shall use the term "subsequence argument" in future to refer to any arguments where we alter Q_k or E_k as above.

As the previous paragraph indicates, the "k" factor on the right-hand side of (3.2) is merely a convenience to simplify subsequence arguments: it could be eliminated as long as one still assumed that $R_k \to \infty$. In fact in each case when we have finished constructing new T-sequences with more desirable properties, our final step before constructing a weight will generally be either to replace E_k by a subset of itself or to decrease R_k so that $|E_k|/|Q_k| = R_k^{-r_2}$ (note that (3.1) remains true under either of these two operations).

We now finish the proof of (i) under the control assumption that, for fixed n, p, q, and f, $WRH_q^{\Omega}(wf)$ is bounded by a constant dependent only on $RH_p^{\Omega}(w)$, the case we shall refer to as the *Special Case.* Let $\{(Q_k, E_k, R_k)\}$ be a T-sequence for f. We write $\sigma_1 = \sqrt{\sigma}$ and choose $E'_k \subset E_k$ so that $a_k \equiv |E'_k|/|Q_k| = R_k^{-r_2}$. Since $x \mapsto px/(p+x)$ is strictly increasing on $(0, \infty)$ and $r_2 < s$, we can choose $\delta > 0$ so small that $pr_2/(p + r_2(1+\delta)^{-2}) < ps/(p+s) = q$. If $w_k = (M\chi_{E'_k})^{1/p(1+\delta)}$, then $\{RH_p^{\mathbf{R}^n}(w_k)\}_{k=1}^{\infty}$ is a bounded sequence.

By Hölder's inequality and the boundedness of M on $L^{1+\delta}$, we get

$$\|w_k\|_{p,\sigma_1 Q_k} \leq \|w_k\|_{(1+\delta)^2 p,\sigma_1 Q_k} = \|M\chi_{E'_k}\|_{1+\delta,\sigma_1 Q_k}^{1/p(1+\delta)} \leq C a_k^{1/p(1+\delta)^2},$$

and, since $f \in WRH_{p'}^{\Omega}$,

$$\|fw_k\|_{1,\sigma_1 Q_k} \leq \|f\|_{p',\sigma_1 Q_k} \|w_k\|_{p,\sigma_1 Q_k} \leq C a_k^{1/p(1+\delta)^2} \|f\|_{1,\sigma Q_k}.$$

Let us write $\rho_k = R_k a_k^{-1/p(1+\delta)^2} = R_k^{(p+r_2(1+\delta)^{-2})/p}$, so $1 < \rho_k \to \infty$ $(k \to \infty)$. Since $fw_k > R_k \|f\|_{1,\sigma Q_k}$ on E'_k,

$$\rho_k \leq C \frac{\min_{E'_k} fw_k}{\|fw_k\|_{1,\sigma_1 Q_k}}.$$

Therefore, $E(C\rho_k, Q_k; fw_k, \sigma_1) \supset E'_k$ and so

$$\forall k: \quad \frac{|E(C\rho_k, Q_k; fw_k, \sigma_1)|}{|Q_k|} \geq a_k = \rho_k^{-pr_2/(p+r_2(1+\delta)^{-2})} > \rho_k^{-q}. \tag{3.3}$$

It follows that $\{WRH_q^\Omega(fw_k)\}$ cannot be a bounded sequence since 2.1(iv) and the comments preceding that lemma would then imply that $R^q|E(R,Q;fw_k,\sigma_1)|/|Q|$ tends to zero as $R \to \infty$, uniformly over all cubes Q and all k. Since $\{WRH_q^\Omega(fw_k)\}$ is unbounded but $\{RH_p^\Omega(w_k)\}_{k=1}^\infty$ is bounded, we have arrived at a contradiction to our control assumption.

We now wish to create a single weight w so as to eliminate the control assumption. Let $S = \{\sigma Q_k\}_{k=1}^\infty$. If the cubes in S are pairwise disjoint for any $\sigma > 1$, Lemma 2.3(i) produces such a weight by patching together the weights w_k above. Suppose therefore that the cubes in S are not pairwise disjoint. We call σQ_j *isolated* (with respect to S) if σQ_j intersects only finitely many other cubes in S. If S has infinitely many isolated cubes, we can construct a pairwise disjoint subsequence (let k_1 be the index of the first isolated cube, and inductively let k_{j+1} be the first index larger than k_j of an isolated cube which does not intersect any of the previous cubes in the subsequence). Lemma 2.3(i) can then be applied to the associated T-subsequence.

Thus we may assume that there are only finitely many isolated cubes in S, and that the same is true for any subsequence of S. We may in fact assume that the cubes σQ_k are pairwise intersecting. To justify the latter assumption, we need to prove that there is a subsequence $\{\sigma Q_{k_j}\}_{j=1}^\infty$ of S with this property. To see this, let k_1 be the index of the first non-isolated cube. Eliminating σQ_{k_1} and all cubes which do not intersect σQ_{k_1}, we are left with a subsequence which we will name $\{P_k\}$. There are only finitely many isolated cubes in this subsequence, so suppose that P_l is the first non-isolated cube. We let k_2 be the index in the original sequence of the cube P_l, and then eliminate from this subsequence P_l and all cubes which do not intersect it. Continuing this process, we get the required subsequence.

We now divide the problem into three main cases, characterised by whether $l(Q_k)$ remains roughly constant, tends to 0, or tends to ∞ (by taking a subsequence, we can always get one of these types). We assume that $\sigma' = 10$, $\sigma = 2$ and $\sigma_1 = \sqrt{\sigma}$ throughout.

Case 1: $\{l(Q_k)\}$ *is bounded above and below.*

Since the cubes $\{2Q_k\}$ are pairwise intersecting, it follows that $\{Q_k\}$ is compactly supported in Ω. By choosing a subsequence if necessary, we can assume that the vertices of Q_k converge to the corresponding vertices of some fixed cube Q_∞. Choosing $t \in (1,2)$ so close to 1 that $f(\sigma t Q_\infty)/2 \leq f(\sigma Q_\infty) \leq 2f((\sigma/t)Q_\infty)$, we may assume that $(1/t)Q_\infty \subset Q_k \subset tQ_\infty$ for all k. By a subsequence argument, we may assume that all our cubes Q_k are the same 5-dilatable cube $Q = tQ_\infty$. We may also assume that $R_{k+1}/R_k > 4^{1/r_2}$ for all k, and normalise f so that $\|f\|_{1,\sigma Q} = 1$. Let $E'_k \subset E_k$ be such that $|E'_k|/|Q| = R_k^{-r_2}$, and let $D_k = E'_k \setminus \bigcup_{j>k} E'_j$ (so that $|D_k| > |E'_k|/2$).

We choose $\delta > 0$ so small that $pr_2/(p + r_2(1+\delta)^{-3}) < q$. Letting $r_3 = r_2(1+\delta)^{-2}$, and $v = \sum_{k=1}^{\infty} R_k^{r_3} \chi_{D_k}$, we note that $w = (Mv)^{1/p(1+\delta)} \in RH_p^{\Omega}$. Also,

$$\|w\|_{p,\sigma_1 Q} \le \|w\|_{p(1+\delta)^2,\sigma_1 Q} = \|Mv\|_{1+\delta,\sigma_1 Q}^{1/p(1+\delta)} \le C\|v\|_{1+\delta,\sigma_1 Q}^{1/p(1+\delta)} \le C$$

Since $fw > R_k^{1+r_3/p(1+\delta)} \equiv \rho_k$ on D_k, it follows as in the Special Case that

$$2|E(C\rho_k, Q; fw, \sigma_1)|/|Q| \ge 2|D_k|/|Q| > R_k^{-r_2} = \rho_k^{-r_2 p/(p+r_2(1+\delta)^{-3})} > \rho_k^{-q}.$$

Lemma 2.1 now gives the required contradiction.

Case 2: $l(Q_k) \to 0\,(k \to \infty)$.

Clearly, we may assume that $l(Q_{k+1}) < l(Q_k)/100$. Although $\sigma = 2$, we may actually assume that the cubes $\{\sigma_2 Q_k\}$ are pairwise intersecting, where $\sigma_2 = 11/10$. This is because a T-sequence with parameters σ, σ' has a subsequence which is a T-sequence with parameters σ_2, σ' (simply decrease each R_k by a factor σ^n/σ_2^n, and recover (3.2) by taking the subsequence consisting of every Nth triple for appropriately large N). As before, one can extract either a pairwise disjoint or a pairwise intersecting subsequence of the associated subsequence of $\{\sigma_2 Q_k\}$; we have, of course, already handled the former case.

Since the cubes $\{\sigma_2 Q_k\}$ are now assumed to be pairwise intersecting, it follows that $\sigma_3 Q_k \supset 2(\sigma_3 Q_{k+1})$, where $\sigma_3 = 6/5$. Also, $\{(\sigma_3 Q_{Nk}, E_{Nk}, R_{Nk})\}_{k=1}^{\infty}$ is a T-sequence with parameters σ/σ_3 and σ'/σ_3, where N is any integer greater than σ_3^n. Thus if we take σ/σ_3 and σ'/σ_3 as the parameters of our original T-sequence, we may assume that $2Q_{k+1} \subset Q_k$, for all $k \ge 1$.

We may further assume, by induction, that $l(Q_k)$ decreases fast enough that $|E_k \backslash 2Q_{k+1}| \ge |E_k|/2$. Replacing E_k by $E_k \backslash 2Q_{k+1}$, a subsequence argument gives us a new T-sequence; thus we may assume that E_k and $2Q_{k+1}$ are disjoint. As before, we now choose a subset E_k' of E_k such that $|E_k'|/|Q_k| = R_k^{-r_2}$. We apply Lemma 2.3(ii) to produce a weight w such that $w(x) \approx c_k \left(M\chi_{E_k'}(x)\right)^{1/p(1+\delta)}$ for $x \in Q_k \backslash 2Q_{k+1}$. Clearly, Q_k contains a cube P_k disjoint from $2Q_{k+1}$ and of sidelength at least $l(Q_k)/3$. Let us write $\sigma_4 = \sqrt{\sigma/\sigma_3}$. Using the doubling property of w^p, we see that

$$\begin{aligned} \|w\|_{p,\sigma_4 Q_k} &\le C\|w\|_{p,P_k} \le C\left\| c_k \left(M\chi_{E_k'}\right)^{1/p(1+\delta)} \right\|_{p,P_k} \\ &\le Cc_k \|M\chi_{E_k'}\|_{1+\delta,P_k}^{1/p(1+\delta)} \le Cc_k \left(|E_k'|/|Q_k|\right)^{1/p(1+\delta)^2}. \end{aligned}$$

This case can now be finished as in the Special Case.

Case 3: $l(Q_k) \to \infty\,(k \to \infty)$.

As in Case 2, we may assume that $2Q_k \subset Q_{k+1}$ for all $k \ge 1$ if we change the parameters of the T-sequence to σ/σ_3 and σ'/σ_3. We write $\sigma_4 = \sqrt{\sigma/\sigma_3}$. If, for all $k \geq 1$, $|E_{k+1} \backslash 2Q_k| \geq |E_{k+1}|/2$, we simply apply Lemma 2.3(ii), as in Case 2. If we cannot make such an assumption, even by taking a subsequence, there must exist k_0 such that for $k \geq k_0$, $|E_k \backslash 2Q_{k_0}| < |E_k|/2$. Replacing E_k by $E_k \cap 2Q_{k_0}$, a subsequence argument allows us to assume that $E_k \subset Q_1$, for all $k \geq 1$; in particular $|E_k|$ is bounded above. By again taking a subsequence, we may assume that $\{|E_k|\}$ is either bounded away from 0, or has limit 0.

Subcase 3a: $|E_k| \geq \epsilon > 0$ *for all* k.

We normalise f so that $\|f\|_{1,\sigma Q_1} = 1$. Since $h(t) = |E(t, Q_1; f, \sigma)|$ is decreasing and upper semicontinuous on $(0, \infty)$, there exists $t_0 > 0$ such that

$$h(t) \geq \epsilon, \qquad t \leq t_0$$
$$h(t) < \epsilon, \qquad t > t_0.$$

Writing $E = E(t_0, Q_1; f, \sigma)$, we have $t_0 = \inf_{x \in E} f(x) \geq \inf_{x \in E_k} f(x)$ for all k, and so by a subsequence argument, we may assume that $E_k = E$ for all k. Letting $w(x) = (M\chi_E(x))^{1/p(1+\delta)}$, we see that

$$\|w\|_{p,\sigma_4 Q_k} \leq C a_k^{1/p(1+\delta)^2}, \qquad \text{where } a_k = |E|/|Q_k|.$$

We replace R_k by $R'_k = a_k^{-1/r_2}$, noting that $\{R'_k\}$ is a sequence tending to infinity and that $R'_k \leq R_k$ (so (3.1) remains true when we replace R_k by R'_k). The proof is finished as in the Special Case (w, E, and R'_k play the roles of w_k, E'_k and R_k respectively).

Subcase 3b: $|E_k| \to 0$ $(k \to \infty)$.

Here we may assume that $|E_{k+1}| < |E_k|/4$ for all k. Let us choose $\delta > 0$ so small that $pr_2/(p + r_2(1+\delta)^{-3}) < q$. We define $D_k = E_k \backslash \bigcup_{j>k} E_j$, $v = \sum_{k=1}^{\infty} (|Q_1|/|D_k|)^{1/(1+\delta)^2} \chi_{D_k}$, and $w = (Mv)^{1/p(1+\delta)}$. Clearly $|D_k| \geq 2|E_k|/3$ and so $|D_k| > 3|D_{k+1}|$. We write $a_k = |D_k|/|Q_k|$ and replace R_k by $R'_k = a_k^{-1/r_2}$; again $R'_k < R_k$ (for $k > 1$) and $R'_k \to \infty$ $(k \to \infty)$. Therefore

$$\|w\|_{p,\sigma_4 Q_k}^{p(1+\delta)^2} \leq \|v\|_{1+\delta,\sigma_4 Q_k}^{1+\delta} \leq C \left(\frac{1}{|Q_k|} \sum_{j=1}^{\infty} |Q_1|^{1/(1+\delta)} |D_j|^{\delta/(1+\delta)} \right)$$
$$\leq C \frac{|Q_1|}{|Q_k|} \left(\frac{|D_1|}{|Q_1|} \right)^{\delta/(1+\delta)} \leq C \frac{|Q_1|}{|Q_k|}.$$

If $x \in D_k$, we get as before that

$$\begin{aligned} f(x)w(x) &> R'_k (|Q_1|/|D_k|)^{1/p(1+\delta)^3} \|f\|_{1,\sigma_3 Q_k} \\ &\geq C R'_k (|Q_1|/|D_k|)^{1/p(1+\delta)^3} (|Q_k|/|Q_1|)^{1/p(1+\delta)^2} \|fw\|_{1,\sigma_4 Q_k} \\ &\geq C R'_k (|Q_k|/|D_k|)^{1/p(1+\delta)^3} \|fw\|_{1,\sigma_4 Q_k}. \end{aligned}$$

Letting $\rho_k = R'_k a_k^{-1/p(1+\delta)^3} = a_k^{-[p+r_2(1+\delta)^{-3}]/pr_2}$, we see that for $k \geq 2$,

$$|E(C\rho_k, Q_k; fw, \sigma_4)| \geq a_k = \rho_k^{-pr_2/(p+r_2(1+\delta)^{-3})} \geq \rho_k^{-q}.$$

Lemma 2.1 now gives the required contradiction. □

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