CHARACTERS, BILINEAR FORMS AND SOLVABLE GROUPS

JOHN C. MURRAY AND GABRIEL NAVARRO

1. Introduction

Let G be a finite group with irreducible 2-Brauer characters $\mathrm{IBr}(G)$. The theory of real valued characters and self-dual G-modules over a field of characteristic 2 admits some remarkable improvements if G is restricted to being solvable.

Theorem 1. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued and non-trivial. Then there exists (U, δ) such that $U \subseteq G$, $\delta \in \mathrm{IBr}(U)$, $\delta^G = \varphi$, δ is real valued and $\delta(1)_2 = 2$. Moreover, the Sylow 2-subgroups of U are determined by φ up to G-conjugacy.*

We use the Isaacs nucleus of a lift of φ to prove the existence of our 'extended nucleus'. The uniqueness part on the Sylow 2-subgroups of U lies much deeper and its proof relies on the new theory of **symmetric vertices** developed by the first author.

Now we turn our attention to Frobenius-Schur indicators. For a character χ of G the indicator $\nu(\chi)$ is the average value of $\chi(g^2)$ for $g \in G$. If χ is irreducible then $\nu(\chi)$ takes one of the values $+1, -1, 0$, as χ is afforded by a real representation or is real-valued but not afforded by a real representation or is not real-valued, respectively. We use the extended nucleus to answer an old question of W. Willems [W91, p518].

Theorem 2. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued. Then G has a real representation whose character lifts φ.*

Next recall that the decomposition numbers $d_{\chi\varphi}$ are given by

$$\chi(g) = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(g), \quad \text{for all odd order } g \in G.$$

Then $\Phi_\varphi := \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$ is called the principal indecomposable character of φ. It is known that Φ_φ vanishes on all elements of even order. In [R89] G. R. Robinson used this to show that $\nu(\Phi_\varphi) \geq 0$. This result is peculiar to $p = 2$.

Theorem 3. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued and non-trivial. Let (U, δ) be an extended nucleus and let (W, γ) be a nucleus of φ. Suppose that $U \backslash W$ contains an involution t. Then $\langle \Phi_\varphi, 1_{C(t)}^G \rangle > 0$ and thus $\nu(\Phi_\varphi) > 0$.*

Date: 26 August 2015.

Key words and phrases. Finite groups, solvable groups, Ordinary characters, Brauer characters, Frobenius-Schur indicator, bilinear forms.

2. Extended Nucleus

As usual $\mathrm{Irr}(G)$ denotes the ordinary irreducible characters of G. Also χ^* denotes the restriction of a character χ to the odd order elements of G.

Let k be a field of characteristic 2 and suppose that S is a non-trivial simple self-dual kG-module. Fong's Lemma asserts that S affords a G-invariant non-degenerate symplectic bilinear form which is unique up to a non-zero scalar. As a consequence, every non-trivial real valued irreducible Brauer character of G has even degree.

Suppose that $\varphi \in \mathrm{IBr}(G)$ and G is solvable. The Fong-Swan theorem asserts that there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^* = \varphi$. Let $H \subseteq G$ be a Hall $2'$-subgroup of G. So $|H|$ is odd and $|G : H|$ is even, and every odd order subgroup of G is contained in a conjugate of H. For the given φ and χ, we say that $\psi \in \mathrm{Irr}(H)$ is a Fong character of χ if $\psi(1)$ is minimal such that $\langle \chi_H, \psi\rangle \neq 0$. In that case it is known that $\langle \chi_H, \psi\rangle = 1$, $\psi(1) = \chi(1)_{2'}$ and ψ^G is the principal indecomposable character of G corresponding to φ.

Lemma 4. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is non-trivial and real valued. Then there is $U \subseteq G$ and a real valued $\delta \in \mathrm{IBr}(U)$ such that $\delta^G = \varphi$ and $\delta(1)_2 = 2$.*

Proof. In [I84] I. M. Isaacs constructed for each $\chi \in \mathrm{Irr}(G)$ a nucleus (W, γ); here $W \subseteq G$ and $\gamma \in \mathrm{Irr}(W)$ is the product of a 2-special character and a $2'$-special character and satisfies $\gamma^G = \chi$. His construction uniquely determines (W, γ) up to G-conjugacy. By definition $\mathrm{B}_{2'}(G)$ is the set of all χ for which γ is $2'$-special. Isaacs showed that $\mathrm{B}_{2'}(G)$ gives a canonical set of lifts for the irreducible Brauer characters of G.

Let $\chi \in \mathrm{B}_{2'}(G)$ with $\chi^* = \varphi$ and let (W, γ) be a nucleus of χ. Then $\bar\chi$ belongs to $\mathrm{B}_{2'}(G)$ as $\bar\chi$ has nucleus $(W, \bar\gamma)$ and $\bar\gamma$ is $2'$-special. Moreover $\bar\chi^* = \bar\varphi = \varphi = \chi^*$. So $\bar\chi = \chi$. On the other hand, γ is non-trivial as φ is non-trivial, and $\gamma(1)$ is odd as γ is $2'$-special. So $\bar\gamma \neq \gamma$, using Fong's Lemma.

Now $(W, \bar\gamma)$ is G-conjugate to (W, γ) as both are nuclei of χ, and $\mathrm{N}_G(W, \gamma) = W$ as $\gamma^{\mathrm{N}_G(W)}$ is irreducible. So the set stabilizer U of $\{\gamma, \bar\gamma\}$ in $\mathrm{N}_G(W)$ satisfies $|U : W| = 2$. It is clear that $\eta = \gamma^U$ is a real valued irreducible character of U with $\eta(1)_2 = 2$. Now set $\delta = \eta^*$, and notice that (U, δ) satisfies what is required. $\square$

Our next result proves a precise form of Theorem 2, thus answering Willems question:

Theorem 5. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued. Let $\chi \in \mathrm{B}_{2'}(G)$ be the Isaacs canonical lift of φ. Then $\nu(\chi) = +1$.*

Proof. We may assume that φ is non-trivial. Let (W, γ) and (U, η) be as in the previous lemma. So $|U : W| = 2$, γ is $2'$-special, $\gamma^U = \eta$ and $\eta_W = \gamma + \bar\gamma$. Now $U = W\langle u\rangle$ where $u \in U \backslash W$ and $u^2 \in W$. We can and do assume that u is a 2-element. Set $C = \langle u\rangle$ and $D = \langle u^2\rangle$, so that $U = WC$ and $C \cap W = D$.

We claim that $\langle \gamma_D, 1_D\rangle$ is odd. To show this, we may assume that $D \neq 1$. Let ζ generate the cyclic group $\mathrm{Irr}(D)$ and set $q = |D|$. Then the rational characters in $\mathrm{Irr}(D)$

are $\zeta^0 = 1_D$ and $\zeta^{q/2}$. Now γ is 2-rational as it is $2'$-special. So γ_D is rational and hence

$$\gamma_D = m_0\zeta^0 + m_{q/2}\zeta^{q/2} + \sum_{i=1}^{q/2-1} m_i(\zeta^i + \bar{\zeta}^i)\,, \quad \text{for non-negative integers } m_i.$$

Then clearly $\det\gamma(u^2) = (-1)^{m_{q/2}}$. But $o(\gamma)$ is odd, as γ is $2'$-special. So $m_{q/2}$ is even. Then $\langle\gamma_D, 1_D\rangle = m_0 \equiv \gamma(1) \pmod 2$. The claim follows, as $\gamma(1)$ is odd.

The previous paragraph implies that $\langle\eta, 1_C^U\rangle$ is odd, as

$$\langle\eta_C, 1_C\rangle = \langle(\gamma^U)_C, 1_C\rangle = \langle\gamma_D, 1_D\rangle.$$

As 1_C^U is afforded by an $\mathbb{R}$-representation of U, this implies that η is afforded by an $\mathbb{R}$-representation of U. So finally $\chi = \eta^G$ is afforded by an $\mathbb{R}$-representation of G. □

Note that it can easily happen that a real irreducible Brauer character of a solvable group has a lift to an ordinary character with Frobenius-Schur indicator -1. For example, let G be the non-abelian group $C_3 \rtimes C_4$ and let $\varphi \in \mathrm{IBr}(G)$ with $\varphi(1) = 2$. Then $\Phi_\varphi = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ are real valued and $\chi_1^* = \chi_2^* = \varphi$. Now $\nu(\Phi_\varphi) = 0$ (see [M06, Theorem 2]). So we can choose notation so that $\nu(\chi_1) = +1$ and $\nu(\chi_2) = -1$.

3. Symmetric Vertices and Extended Nucleus

For the moment k is a field of arbitrary characteristic p. Let $H \subseteq G$. Following [H54] a kG-module M is H-projective if it is a direct summand of an induced module $\mathrm{Ind}_H^G(L)$, for some kH-module L. Suppose that M is indecomposable. Following [G59] a vertex of M is a minimal $V \subseteq G$ such that M is V-projective. The vertices of M are p-subgroups of G which are determined up to G-conjugacy. Now a V-source of M is an indecomposable kV-module Z such that M is a direct summand of $\mathrm{Ind}_H^G(Z)$. Then Z is a direct summand of $\mathrm{Res}_V^G(M)$, and Z is uniquely determined by M and V up to $N_G(V)$-conjugacy.

Recall that the dual of a left kG-module M is the left kG-module $M^* = \mathrm{Hom}_{kG}(M, k)$. Here if $f : M \to k$ and $g \in G$, we set $(gf)(m) := f(g^{-1}m)$, for all $m \in M$. Now $M \cong M^*$ as kG-modules if and only if there exists a G-invariant non-degenerate bilinear form $b : M \times M \to k$. We say that b is symmetric if $b(m_1, m_2) = b(m_2, m_1)$, alternating if $b(m_1, m_2) = -b(m_2, m_1)$ and symplectic if $b(m_1, m_1) = 0$, for all $m_1, m_2 \in M$. If $p \neq 2$ alternating is the same as symplectic and no symplectic form is symmetric. If $p = 2$ alternating is the same as symmetric and all symplectic forms are symmetric but not all symmetric forms are symplectic.

Let (L, c) be a symmetric kH-module. Now $\mathrm{Ind}_H^G(L) = \sum_{gH} g \otimes L$ as k-vector spaces, where $g \otimes L$ is a k^gH-module. The obvious isomorphism $H \cong {}^gH$ maps L to $g \otimes L$. So $g \otimes L$ inherits a ${}^gH$-invariant non-degenerate form ${}^gc$ from c. The induced symmetric kG-module $\mathrm{Ind}_H^G(L, c)$ is the orthogonal direct sum of the symmetric k-spaces $(g \otimes L, {}^gc)$.

Following [M15] a symmetric kG-module (M, b) is H-projective if (M, b) is an orthogonal direct summand of $\mathrm{Ind}_H^G(L, c)$, for some symmetric kH-module (L, c). Moreover a symmetric vertex of M is a minimal $T \subseteq G$ such that there exists a T-projective symmetric kG-module (M, b). Analogous concepts exist for alternating kG-modules.

For the remainder of this section k is a perfect field of characteristic 2 which is a splitting field for all subgroups of G. We simplify our exposition by referring to both symplectic and non-symplectic symmetric forms as symmetric forms. In practice symplectic forms are more important than non-symplectic symmetric forms, because the isometry group of a symmetric form is closely related to a symplectic group.

Example 6. *There is a unique non-trivial simple kD_{12}-module, where D_{12} is the dihedral group of order 12. Its projective cover P affords a 2-dimensional space of D_{12}-invariant symmetric bilinear forms. It can be shown that each non-central C_2-subgroup of D_{12} is a symmetric vertex of P. As there are two D_{12}-conjugacy classes of such subgroups, this shows that symmetric vertices are not unique determined up to G-conjugacy.*

However, the first author proved the following result in [M15]:

Proposition 7. *The symmetric vertices of a self-dual simple kG-module S are uniquely determined up to G-conjugacy. Let b be a symmetric form and let (V, Z) be a vertex-source pair of S. Then S has a symmetric vertex $T \supseteq V$ and exactly one of (i) or (ii) holds:*

(i) *$T = V$ and b is non-degenerate on a submodule of $\mathrm{Res}^G_V(S)$ isomorphic to Z. Moreover $\mathrm{Ind}^G_V(Z) \cong S \oplus Q$, where Q has no summands isomorphic to S.*

(ii) *$|T : V| = 2$ and $\mathrm{Ind}^T_V(Z)$ affords a non-degenerate T-invariant symmetric form c. For any such form c, (S, b) is an orthogonal direct summand of $\mathrm{Ind}^G_T(\mathrm{Ind}^T_V(Z), c)$.*

We shall see in Lemma 9 that only (ii) occurs when G is solvable and S is non-trivial.

L. Puig has shown that if G is solvable then the source Z of a simple module S is an endo-permutation module constructed from tensor products of endo-trivial modules of quotients of a vertex (c.f. [Mz06, Abstract]). As a consequence of the classification of torsion endo-trivial modules for p-groups and [CT00], the sources are self-dual unless a vertex has a generalized quaternion quotient. We present an example of a solvable group with a simple self-dual module which has a non self-dual source, as this seems to be a relatively uncommon phenomenon:

Example 8. *Let E be an extra-special group of order 27 and exponent 3. Then $\mathrm{Aut}(E) \cong \mathrm{GL}(2, 3)$ has a Sylow 2-subgroup T which is semi-dihedral of order 16. Set $G = E \rtimes T$. The centralizer of $Z(E)$ in T is a quaternion group V of order 8. Let k be a field extension of $\mathbb{F}_4$. Then kE has a faithful 3-dimensional module, which extends to a simple $kE \rtimes V$-module M. Now $M^T = M^* \not\cong M$. So $S = \mathrm{Ind}^G_{E \rtimes V}(M)$ is a self-dual simple kG-module with vertex V. Moreover S has V-source $Z := \mathrm{Res}^{E \rtimes V}_V(M)$. As Z is a 3-dimensional endo-trivial kV-module, Z is not self-dual* [CT00, p322]. *So S has symmetric vertex T.*

Theorem 1 is a consequence of our next lemma and the uniqueness of symmetric vertices proved in Proposition 7.

Lemma 9. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is non-trivial and real valued. Let $U \subseteq G$ and $\delta \in \mathrm{IBr}(U)$ be such that δ is real valued, $\delta^G = \varphi$ and $\delta(1)_2 = 2$. Let S be the simple kG-module whose Brauer character is φ. Then each Sylow 2-subgroup T of U is a symmetric vertex of S.*

Proof. Let (W,γ) be the Isaacs nucleus of the lift of δ in $\mathrm{B}_{2'}(U)$, let S_U be the simple kU-module with Brauer character δ and let S_W be the simple kW-module with Brauer character γ^*. Recall that $S_W \not\cong S_W^*$ as $\gamma(1)$ is odd. Let (V,Z) be a vertex source pair of S_W. Then it is clear that (V,Z) is a vertex source pair of S and S_U.

We claim that V is not a symmetric vertex of S. For otherwise $Z \cong Z^*$ by the first statement in Proposition 7(i). So Z is a V-source of S_W^*. In particular S_W and S_W^* are non-isomorphic components of $\mathrm{Ind}_V^W(Z)$. Now $\mathrm{Ind}_W^G(S_W) \cong S \cong S^* \cong \mathrm{Ind}_W^G(S_W^*)$. So S occurs at least twice as a direct summand of $\mathrm{Ind}_V^G(Z)$. This contradicts the second statement in Proposition 7(i), which proves our claim.

We can apply the previous paragraph to S_U. So V is not a symmetric vertex of S_U. Then by Proposition 7(ii), S_U has a symmetric vertex $T \supseteq V$ with $|T:V| = 2$. Now V is a Sylow 2-subgroup of W, as $\dim(S_W)$ is odd. But $|U:W| = 2$. So T is a Sylow 2-subgroup of U. Now let b_U be a symmetric form on S_U. Then $(S,b) \cong \mathrm{Ind}_U^G(S_U,b_U)$ as b is unique up to isometry. Moreover b_U is T-projective. So it follows from the transitivity of induction of forms that b is T-projective. Since $|T:V| = 2$, we deduce that T is a symmetric vertex of S. □

4. Projective Indecomposable Modules and Orthogonal Forms

Temporarily let k be a field of arbitrary characteristic p. The study of bilinear and quadratic forms on projective kG-modules has attracted some interest. There are ring-theoretic criteria for a projective indecomposable kG-module to be of quadratic type (have a non-degenerate G-invariant quadratic form). These are due to Landrock and Manz [LM92] for $p \neq 2$, and to Gow and Willems [GW93] for $p = 2$.

Recall that the Jacobson radical $J(kG)$ of kG is the annihilator of all simple kG-modules and the contragredient map ^{o} is the k-algebra involutary anti-automorphism of kG such that $g^o = g^{-1}$, for all $g \in G$.

Proposition 10 (Landrock-Manz)**.** *Suppose that $p \neq 2$ and P is a projective indecomposable kG-module. Then P is of quadratic type if and only if there is a primitive idempotent e in kG such that $P \cong kGe$ and $e^o = e$.*

From now on k is a perfect field of characteristic $p = 2$. From [GW93], if P is the projective cover of a non-trivial simple kG-module then each G-invariant symmetric form on P is the polarization of a G-invariant quadratic form on P. In particular each such form is symplectic. Now a primitive idempotent $e \in kG$ satisfies $e^o = e$ if and only if kGe is the projective cover of the trivial kG-module. So Proposition 10 is wrong for $p = 2$, and is replaced by:

Proposition 11 (Gow-Willems)**.** *Suppose that e is a primitive idempotent in kG. Then kGe is of quadratic type if and only if there is an involution $t \in G$ such that $e^oe^t \notin J(kG)$.*

If $e^oe^t \notin J(kG)$ there is a unique idempotent $f \in kG$ such that $kGe = kGf$ and $f^o = f^t$.

Parts of this result are only implicit in [GW93, Section 3].

G. R. Robinson showed in [R89] that if Φ is a principal indecomposable character of G then $\nu(\Phi)=\sum_t\langle\Phi,1_{C_G(t)}^G\rangle$ where t ranges over 1 and the conjugacy classes of involutions in G. The first author showed in [M06, Corollaries 5.2 and 6.5]:

Lemma 12. *Suppose that e is a primitive idempotent in kG and $t\in G$ is an involution such that $e^oe^t\notin J(kG)$. Let Φ be the principal indecomposable character of kGe. Then $\langle\Phi,1_{C_G(t)}^G\rangle>0$. In particular $\nu(\Phi)>0$, if kGe has a quadratic geometry.*

For G solvable, we aim to directly relate the Gow-Willems criterion to the extended nucleus and symmetric vertex of the corresponding simple modules. We begin with a very general remark, which holds for an arbitrary field k:

Lemma 13. *Suppose that N is a normal subgroup of G. Then $J(kN)=J(kG)\cap kN$.*

Proof. Let S be a simple kG-module. Then $\mathrm{Res}_N^G(S)$ is semi-simple, by Clifford's theorem. So $J(kN)\subseteq J(kG)\cap kN$. Conversely, let S_N be a simple kN-module. Then $\mathrm{Res}_N^G\mathrm{Ind}_N^G(S_N)=\sum_{gN\subseteq G}S_N^g$ by Mackey's formula. This implies that S_N is a direct summand of $\mathrm{Res}_N^G(S)$, for some simple kG-module S. So $J(kN)\supseteq J(kG)\cap kN$. □

We also need a result from [GW95]:

Lemma 14. *Suppose that (M,b) is a symmetric kG-module and $M=M_1\dot{+}\dots\dot{+}M_t$ is a decomposition of M as an internal direct sum of indecomposable kG-modules M_i. Then for each i, either b is non-degenerate on M_i or there exists $j\neq i$ such that $M_j\cong M_i^*$ and b is non-degenerate on $M_i\dot{+}M_j$.*

Let $P_G(M)=P(M)$ denote the projective cover of a kG-module M. Theorem 3 is a consequence of Lemmas 9 and 12 and our next result:

Theorem 15. *Suppose G is solvable and S is a self-dual simple kG-module with a vertex and symmetric vertex $V\subseteq T$. Then $P(S)$ is of quadratic type if and only if $T:V$ splits.*

Proof. If S is trivial, then $T=V$ and it is easy to see that $P(S)$ has a quadratic geometry. So from now on S is non-trivial. Let φ be the Brauer character of S and let $\chi\in\mathrm{B}_{2'}(G)$ with $\chi^*=\varphi$. Also let (W,γ) and (U,δ) be as in Lemma 4. So $\delta(1)_2=2$ and $\delta^G=\varphi$.

Let S_U be the self-dual simple kU module whose Brauer character is δ. As $\mathrm{Ind}_U^G(S_U)=S$, Frobenius-Nakayama reciprocity implies that $\mathrm{Res}_U^G(P(S))=P(S_U)\oplus Q$, where no component of Q is isomorphic to $P(S_U)^*\cong P(S_U)$.

Suppose first that $P(S)$ is of quadratic type. Then $P(S_U)$ is of quadratic type, by the previous paragraph and Lemma 14. Now $\mathrm{Res}_W^U(S_U)=S_W\oplus S_W^*$, where S_W is the simple kW-module whose Brauer character is γ^*. Let e be a primitive idempotent in kW such that $kWe\cong P(S_W)$. Then e is still primitive in kU and indeed $kUe\cong P(S_U)$. So according to Proposition 11, there is an involution $t\in U$ such that $e^oe^t\notin J(kU)$.

We claim that $t\notin W$. For suppose otherwise. Then $e^oe^t\in kW$. But Lemma 13 implies that $e^oe^t\notin J(kW)$. So $P(S_W)\cong kWe$ is of quadratic type and in particular $S_W\cong S_W^*$.

This contradiction proves the claim. We have shown that U splits over W. So T splits over V, by Lemma 9.

Suppose now that T splits over V. Let t be any involution in $T\backslash V$ and let H be a Hall $2'$-subgroup of G such that $H \cap W$ is a Hall $2'$-subgroup of W. As $\chi = \gamma^G$ we have

$$\chi(1)_{2'} = [G : W]_{2'}\gamma(1) = |H : H \cap W|\gamma(1) = (\gamma_{H\cap W})^H(1).$$

Moreover $\langle \chi, (\gamma_{H\cap W})^G\rangle \geq \langle \gamma, (\gamma_{H\cap W})^H\rangle \geq 1$. So $\gamma_{H\cap W}$ is a Fong character for γ and $(\gamma_{H\cap W})^H$ is a Fong character for χ. Then $\Phi_\delta = (\gamma_{H\cap W})^U$ and $\Phi_\varphi = (\gamma_{H\cap W})^G$ are the principal indecomposable characters of S_U and S, respectively. In particular $\Phi_\delta^G = \Phi_\varphi$. It follows from this that $\mathrm{Ind}_U^G(P(S_U)) = P(S)$. So to complete the proof we need only show that $P(S_U)$ is of quadratic type.

We can and do assume that $U = G$, $S_U = S$ and thus $|G : W| = 2$. Set $N = \mathrm{O}_{2'}(G)$.

Suppose first that N acts trivially on S. Set $L = O_{2',2}(G)$ and $\overline{G} = G/L$. Then S can be identified (by deflation) with an irreducible $k\overline{G}$-module. As $k\overline{G}$-module it has vertex $\overline{V}$ and symmetric vertex $\overline{T}$. Now $\overline{t}$ is an involution in $\overline{T}\backslash\overline{V}$ and $|G/L| < |G|$. So by induction on $|G|$ there is a primitive idempotent $\overline{e} \in k\overline{G}$ such that $k\overline{G}\overline{e} \cong P_{\overline{G}}(S)$ and $\overline{e}^{\overline{t}} = \overline{e}^o$.

The map $x^\sigma := tx^ot$, for $x \in kG$, is an involutary k-algebra anti-automorphism of kG. The kernel of the projection map $kG \to k\overline{G}$ is $\mathrm{sp}\{g(1-\ell) \mid g \in G, \ell \in L\}$. It is easy to check that this is σ-invariant. So σ induces the involutary k-algebra anti-automorphism $\overline{x}^\sigma = \overline{t}\overline{x}^o\overline{t}$ on $k\overline{G}$.

Notice that $\overline{e}^\sigma = \overline{e}$. By idempotent lifting [M15, Lemma 2.1] there is a primitive idempotent $e \in kG$ such that $e^\sigma = e^o$ and $\overline{e}$ is the image of e in $k\overline{G}$. Then Proposition 11 implies that $kGe \cong P(S)$ is of quadratic type. This completes the case $N \subseteq \ker(S)$.

Let $\theta \in \mathrm{Irr}(N|\gamma)$. By the work above we may assume that θ is non-trivial. In particular $\theta \neq \overline{\theta}$. Set $m := \langle \chi_N, \theta\rangle = \langle \gamma_N, \theta\rangle$. Then m is odd, as it divides $\gamma(1)$. Let Z be the simple kN-module whose Brauer character is θ. Then Z occurs m times as a direct summand of the semisimple kN-module $\mathrm{Res}_N^G(S)$. So by Frobenius-Nakayama reciprocity, $P(S)$ occurs m times as a direct summand of the projective kG-module $\mathrm{Ind}_N^G(Z)$.

Now $m = |W : \mathrm{N}_W(\theta)|$ is odd. So $\mathrm{N}_W(\theta)$ contains a Sylow 2-subgroup of W. Moreover θ is G-conjugate to $\overline{\theta}$ as both belong to $\mathrm{Irr}(N|\chi)$. So $|\,\mathrm{N}_G(\theta, \overline{\theta}) : \mathrm{N}_G(\theta)| = 2$. As $|G : W| = 2$, it follows that $\mathrm{N}_G(\theta, \overline{\theta})$ contains a Sylow 2-subgroup of G. So we can and do assume that T is a Sylow 2-subgroup of $\mathrm{N}_G(\theta, \overline{\theta})$ and $V = T \cap \mathrm{N}_G(\theta)$. In particular $\theta^t = \overline{\theta}$.

Consider the group $E := N\langle t\rangle$, which is a degree 2-extension of N. Then $\mathrm{Ind}_N^E(Z)$ is a simple kE-module which is self-dual as its Brauer character is θ^E. So it affords a non-degenerate E-invariant symplectic bilinear form which is $\langle t\rangle$-projective. As $P(S)$ occurs with odd multiplicity m in $\mathrm{Ind}_N^G(Z) = \mathrm{Ind}_E^G(\mathrm{Ind}_N^E(Z))$, we deduce that $P(S)$ affords a non-degenerate G-invariant symplectic bilinear form which is $\langle t\rangle$-projective. In particular $P(S)$ is of quadratic type and there is a primitive idempotent $e \in kG$ such that $e^t = e^o$ and $P(S) \cong kGe$. This completes the proof of the theorem. □

5. Acknowledgement

This paper was initiated at the conference 'Representations of Finite Groups' held at the Mathematisches Forschungsinstitut Oberwolfach in April 2015. We thank the institute for its support. N. Mazza alerted us to the significance of the torsion endo-trivial modules of quaternion groups for the sources of simple modules of solvable groups. Following our query, E. C. Dade found a simple self-dual module for a solvable group which has a non self-dual source. We briefly describe this in Example 8, with his kind permission.

References

[CT00] J. F. Carlson, J. Th'evenaz, 'Torsion endo-trivial modules', *Algebr. Represent. Theory* **3** (4) (2000) 303335.

[G59] J. A. Green, 'On the indecomposable representations of a finite group', *Math. Zeit.* 70 (1959) 430–445.

[GW93] R. Gow, W. Willems, 'Quadratic geometries, projective modules and idempotents', *J. Algebra* **160** (1993) 257–272.

[GW95] R. Gow, W. Willems, 'A note on Green correspondence and forms', *Comm. Algebra* **23** (4) (1995) 1239–1248.

[H54] Higman, D. G., 'Modules with a group of operators', *Duke Math. J.* **21** (1954). 369376.

[I84] I. M. Isaacs, 'Characters of π-seperable groups', *J. Algebra* **86** (1984) 98–128.

[LM92] P. Landrock, O. Manz, 'Symmetric forms, idempotents and involutary anti-isomorphisms', *Nagoya Math. J.* **125** (1992) 33–51.

[Mz06] N. Mazza, 'Endo-permutation modules as sources of simple modules', *J. Group Theory* **6** (4) (2003) 477497.

[M06] J. Murray, 'Strongly real 2-blocks and the Frobenius-Schur indicator', *Osaka J. Math.* **43** (1) (2006) 201-213.

[M15] J. Murray, 'Symmetric Vertices for Symmetric Modules in Characteristic 2', submitted for publication, January 2015.

[R89] G. R. Robinson, 'The Frobenius-Schur Indicator and Projective Modules', *J. Algebra* **126** (1989) 252–257.

[W91] W. Willems, 'Duality and forms in representation theory', Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), Progr. Math., 95, Birkhuser, Basel, 1991, 509-520.

Department of Mathematics & Statistics, National University of Ireland Maynooth, IRELAND
E-mail address: John.Murray@maths.nuim.ie

Department of Algebra, University of Valencia, 46100 Burjasot (Valencia), SPAIN
E-mail address: Gabriel.Navarro@uv.es