Problem: Find all polynomials \( f(x) = x^3 + bx^2 + cx + d \), where \( b, c, d \) are real numbers, such that \( f(x^2 - 2) = -f(-x)f(x) \).

Solution: Set \( a := 2 \). Let \( \beta_1, \beta_2, \beta_3 \) be the roots of \( f(x) \). Then the hypothesis implies that

\[
\prod_{i=1}^{3} (x - \sqrt{a + \beta_i})(x + \sqrt{a + \beta_i}) = \prod_{i=1}^{3} (x - \beta_i)(x + \beta_i)
\]

We consider the various possibilities.

Assume first that \( \sqrt{a + \beta_1} = \pm \beta_i \), for all \( i = 1, 2, 3 \). Then each \( \beta_i \) is a root of \( x^2 - x - 2 \). So \( \beta_1 = -1 \) or \( 2 \). It can be checked that this gives four possible polynomials \( f(x) \):

\[
\begin{align*}
(1) \quad & (x + 1)^3, \quad (x + 1)^2(x - 2), \quad (x + 1)(x - 2)^2, \quad (x - 2)^3.
\end{align*}
\]

Assume next that \( \sqrt{a + \beta_1} = \pm \beta_3 \) but \( \sqrt{a + \beta_2} \neq \pm \beta_2 \). Then \( \sqrt{a + \beta_2} = \pm \beta_3 \) and so \( \sqrt{a + \beta_3} = \pm \beta_2 \). Now \( \beta_1 = -1 \) or \( 2 \), as before. Also \( \beta_2 = \beta_3^2 - 2 \) and \( \beta_3 = \beta_2^2 - 2 \). So \( \beta_2, \beta_3 \) are roots of

\[
(x^2 - 2)^2 - x - 2 = (x^2 - x - 2)(x^2 + x - 1)
\]

and hence the two complex conjugate roots \((1 \pm \sqrt{5})/2 \) of \( x^2 + x - 1 \). In this way we obtain two additional possible polynomials \( f(x) \):

\[
(2) \quad (x + 1)(x^2 + x - 1), \quad (x - 2)(x^2 + x - 1).
\]

Finally we consider the case that \( \sqrt{a + \beta_1} \neq \pm \beta_i \), for \( i = 1, 2, 3 \). Then we may choose notation so that (with \( i \) considered mod 3):

\[
\sqrt{a + \beta_{i+1}} = \pm \beta_i;
\]

whence \( \beta_{i+1} = \beta_i^2 - a \), for \( i = 1, 2, 3 \). Thus \( f(x) \) has the three roots:

\[
\beta_1, \quad \beta_1^2 - a, \quad (\beta_1^2 - a)^2 - a = \beta_1^4 - 2a\beta_1^2 + a^2 - a.
\]

Now \( -b \) is the sum of the roots of \( f \). So \( \beta_1 \) is a root of the quartic

\[
g(x) := x^4 + (1 - 2a)x^2 + x + a^2 - 2a + b = 0.
\]

As its roots are distinct, it follows that \( f(x) \) divides \( g(x) \). As \( g(x) \) has zero \( x^3 \) term, its easy to see that the quotient \( g/f \) must be \( x - b \). Thus

\[
(x - b)(x^3 + bx^2 + cx + d) = x^4 + (1 - 2a)x^2 + x + a^2 - 2a + b.
\]

Thus we get

\[
c - b^2 = 1 - 2a, \quad d - bc = 1, \quad -bd = a^2 - 2a + b.
\]

In particular \( c = b^2 - 2a + 1 \) and \( d = bc + 1 \).

Now we expand \( f(x^2 - a) = -f(-x)f(x) \), using \( f(x) = x^3 + bx^2 + cx + d \). The coefficient of \( x^2 \) gives:

\[
b - 3a = 2c - b^2.
\]

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But \( c = b^2 - 2a + 1 \), from the previous paragraph. We deduce that \( 2(b^2 - 2a + 1) = b^2 + b - 3a \), whence \( b^2 - b + (2 - a) = 0 \). As \( a = 2 \), we conclude that \( b = 0 \) or \( 1 \). Thus we get two additional possible polynomials \( f(x) \):

\[
(3) \quad x^3 - 3x + 1, \quad x^3 + x^2 - 2x - 1. 
\]

The set of all possible polynomials \( f(x) \) is contained in (1), (2) and (3).

Notes: It’s slightly harder to classify all \( f(x) \) such that \( f(x^2 - a)/f(x) \) is a polynomial. You can change \( a = 2 \) to any positive real number. For example \( a = 29/16 \) gives ‘unusual’ cubic solutions for \( f(x) \):

\[
x^3 + \frac{3}{4} x^2 - \frac{33}{16} x - \frac{35}{64}, \quad x^3 + \frac{1}{4} x^2 - \frac{41}{16} x - \frac{23}{64}.
\]

The latter is irreducible over \( \mathbb{Q} \), while the former has rational roots

\[
\frac{5}{4}, -\frac{1}{4}, -\frac{7}{4}.
\]

These form a ‘3-cycle’ under the \( x \to (x^2 - 29/16) \) operation:

\[
\frac{5}{4} = (-\frac{7}{4})^2 - 29/16, \quad -\frac{1}{4} = (\frac{5}{4})^2 - 29/16, \quad -\frac{7}{4} = (-\frac{1}{4})^2 - 29/16.
\]

It can be shown that all rational 3-cycles are got from pairs \( s, t \) of coprime integers as:

\[
\frac{s^3 - s^2 t + t^3}{2st(s - t)} \quad \frac{s^3 - 3s^2 t + 2st^2 - t^3}{2st(s - t)} \quad \frac{-s^3 + s^2 t - 2st^2 + t^3}{2st(s - t)}.
\]

Using the idea of cycles, an alternative problem is to find all rational numbers \( \beta \) such that

\[
\beta = \sqrt{29/16 + \sqrt{29/16 + \sqrt{29/16 + \beta}}}. 
\]

The ‘obvious’ approach to such a question is to consider roots of the associated octic polynomial:

\[
x^8 - \frac{(29/2)}{x^7} + (11 \times 17 \times 29/2^6) x^6 - (5 \times 29^2 \times 31/2^8) x^5 + (3 \times 7 \times 29 \times 22291/2^{15}) x^4 - (3 \times 29^2 \times 59 \times 307/2^{17}) x^3 + (3^3 \times 29^2 \times 23173/2^{22}) x^2 - (5 \times 7 \times 13 \times 23 \times 29^3/2^{25}) x - (13 \times 29 \times 397 \times 48371/2^{32}).
\]

Its probably impossible to guess the rational roots of this! However, its not too difficult to see that \((x - 29/16)^2 - x - 29/16\) is a quadratic factor.\( \text{The trick is to look for cubic factors, which will have roots } \beta, \quad \beta^2 - 29/16, \quad (\beta^2 - 29/16)^2 - 29/16. \)

Finally, the problem is based on a rather unusual polynomial identity:

\[
(((x^2 - a)^2 - a^2 - x - a = (x^2 - x - a) f(x, b) f(x, 1 - b),
\]

where \( x, b \) are commuting indeterminates, and \( a = b^2 - b + 2 \) and

\[
f(x, b) = x^3 + bx^2 - (b^2 - 2b + 3)x - (b^3 - 2b^2 + 3b - 1). 
\]

The discriminant of \( f(x, b) \), as polynomial in \( x \), is equal to

\[
\Delta(f(x)) = (4b^2 - 6b + 9)^2 = ((2b^3 - 3)^2 + 27/4)^2.
\]

The extreme situation that \( \Delta(f(x)) = (27/4)^2 \) is minimal occurs when \( b = 3/4 \) and hence \( a = 29/16 \).

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