MODULES FOR CENTRALIZER ALGEBRAS: DRAFT

JOHN MURRAY

1. Introduction

This paper was inspired by Susumu Ariki [2]. Recall that the group ring RS_n of the symmetric group S_n over an integeral domain R is a cellular algebra, in the sense of Graham-Lehrer. Ariki noted that the cell modules for RS_n have combinatorially described RS_ℓ-filtrations. Here we exploit the dual result for Specht modules.

The centralizer algebras in question are $RS_n^{S_\ell}$, where S_ℓ is the centralizer of $\{\ell+1, \ldots, n\}$ in S_n, for $\ell \leq n$. $RS_n^{S_\ell}$ is the set of elements in the group ring RS_n that are fixed under conjugation by elements of S_ℓ. So $RS_n^{S_\ell}$ has a basis consisting of the sums C^+ in RS_n of S_ℓ-orbits C on S_n.

Let $Z(RS_\ell)$ be the centre of the group ring RS_ℓ and let $S'_{n-\ell}$ be the centralizer of $\{1, \ldots, \ell\}$ in S_n. Then $S'_{n-\ell} \cong S_{n-\ell}$ and $S_\ell \times S'_{n-\ell}$ is a subgroup of S_n. The elements $L_i := (1, i) + \cdots + (i-1, i)$ of RS_n, for $i = 2, \ldots, n$, are called Murphy elements. Each of $L_{\ell+1}, \ldots, L_n$ is invariant under the conjugation action of S_ℓ. G. Olshanskii showed that $RS_n^{S_\ell}$ is generated, as R-algebra, by $L_{\ell+1}, \ldots, L_n$, over the subalgebras $Z(RS_\ell)$ and $RS'_{n-\ell}$.

We are interested in the modular representation theory of $RS_n^{S_\ell}$, in particular with the structure and representations of $kS_n^{S_\ell}$, where k is a field of prime characteristic p. To this end, we take (F, R, k) to be a p-modular system, in the sense of Nagao-Tsushima.

The irreducible FS_n-modules are the Specht modules S^λ, one for each partition λ of n. Then the irreducible $FS_n^{S_\ell}$-module are $\mathrm{Hom}_{FS_\ell}(S^\mu, S^\lambda)$, one for each pair of partitions (λ, μ), where μ is a partition of ℓ contained in λ (see below), with corresponding Specht module S^μ for FS_ℓ.

G. Murphy [6] showed that each Specht module S^λ has a basis of simultaneous eigenvectors for the Murphy elements $L_2, \ldots, L_n$. This basis coincides with Young's seminormal basis. In particular, it is easy to write down the representing matrices for the basic transpositions $(1, 2), \ldots, (n-1, n)$ with respect to this basis.

Following the approach pioneered by Okounkov-Vershik (in particular the notion of Gelfand-Tsetlin subalgebras) each of the modules $\mathrm{Hom}_{FS_\ell}(S^\mu, S^\lambda)$ has a basis of simultaneous eigenvectors for the Murphy elements $L_{\ell+1}, \ldots, L_n$. Once again, the representing matrices for the transpositions $(\ell+1, \ell+2), \ldots, (n-1, n)$ in $S'_{n-\ell}$ are known and simple. See [5, 2.3.3] for details about this approach.

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Here we are concerned with constructing an R-form $S^{\lambda/\mu}$ for the module $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$. So $S^{\lambda/\mu}$ is an $RS_n^{S_\ell}$-submodule of $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$, that is free as R-module, and that contains an F-basis of $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$. Our R-form has some very attractive features. It occurs 'naturally' in a combinatorially defined subquotient of S^λ. It has a basis naturally indexed by the standard λ/μ-skew tableau. This basis satisfies analogues of the Garnir relations. Finally, the Murphy elements act by upper-triangular matrices on the standard basis.

The matrices representing the elements of $RS_n^{S_\ell}$ with respect to an R-basis of $S^{\lambda/\mu}$ have entries in R. It follows that we can take their images modulo the unique maximal ideal $J(R)$ of R, to get a representation of $RS_n^{S_\ell}$ (or $kS_n^{S_\ell}$) over k.

The point about doing this *p-modular reduction* is that the composition factors of the p-reductions of the modules $S^{\lambda/\mu}$ include all irreducible $kS_n^{S_\ell}$-modules. Moreover, the *decomposition matrix* (recording the multiplicities of irreducible in the modular reductions) determines the block linkage of $kS_n^{S_\ell}$, and hence the blocks of both this algebra and $RS_n^{S_\ell}$.

Finding the blocks would be trivial if the natural map $Z(RS_n^{S_\ell}) \to Z(kS_n^{S_\ell})$ were surjective. We do not know whether this is the case, at the present time. The analogous result for the centres of degenerate cyclotomic affine Hecke algebras has recently been shown to hold by J. Brundan [1]. H. Ellers and the author [3] computed the blocks of $RS_n^{S_\ell}$ (but not the centre of $kS_n^{S_\ell}$) for $\ell \geq n-3$.

The elements $L_{\ell+1}, \ldots, L_n$ and the basic transpositions in $S'_{n-\ell}$ satisfy the defining relations of a degenerate affine Hecke algebra $\mathcal{H}_{n-\ell}$. So $RS_n^{S_\ell}$ contains a subalgebra that is a quotient of the degenerate affine Hecke algebra. Moreover, $RS_n^{S_\ell}$ is generated by this subalgebra and $Z(RS_\ell)$. The latter algebra is central in $RS_n^{S_\ell}$ and is practically irrelevant to the construction of our modules.

Our construction appear to concur with the construction of skew Specht modules given by Arun Ram in [7]. In particular compare the 'Garnir Relations' given in Theorem 9 below and [7, Theorem 5.5]. There are significant differences. Ram works with modules of the affine Hecke algebra which are called *calibrated* or *completely splittable*. Each of his modules factors through an appropriate cyclotomic quotient. Our modules can be lifted to modules for the degenerate affine Hecke algebras and their cyclotomic quotients.

There is a well-established way of translating results back and forth between the affine Hecke algebra and its degenerate version. However, we have not found a completely satisfactory explanation and description for this process. We prefer the direct approach outlined here, which has not appeared in the literature to date, as far as we can tell.

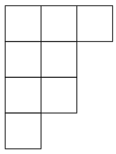
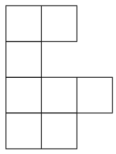
One strength of our approach is that we retain the connection between irreducible modules for $FS_n^{S_\ell}$ and homomorphism spaces between irreducible modules for FS_n and FS_ℓ. This may have applications to the induction and restriction functors between $\mathrm{mod}(RS_\ell)$ and $\mathrm{mod}(RS_n)$. In addition, our approach is elementary, in that all the key concepts can be found in [4]. We hope to mimic the approach of James to the construction of all irreducible kS_n-modules as quotients of certain Specht modules to construct all irreducible modules for $kS_n^{S_\ell}$.

Generally R is an integral domain, F is the field of fractions of R, and $\mathrm{char}(F) = 0$. If M is an R-module, then $M_F := M \otimes_R F$ is the F-vector space obtained by extension of scalars.

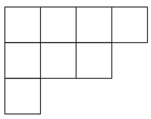
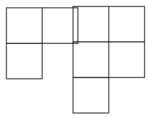
2. Review of the Combinatorics

2.1. A dominance order on tabloids. In this section we make slight adaptations to some notation and results from [4].

Let n be a non-negative integer. A *composition of* n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers whose sum is n. We write $\lambda \models n$. The λ_i are called the *parts* of λ. Each composition λ has an associated diagram $[\lambda]$, which is oriented using the anglophone convention: left and top justified. For example

$[3, 2, 2, 1] =$  $[2, 1, 3, 2] =$ 

The *transpose diagram* $[\lambda]'$ is obtained by reflecting $[\lambda]$ in its main diagonal. It need not be the diagram of a composition. Examples:

$[3, 2, 2, 1]' =$  $[2, 1, 3, 2]' =$ 

The *dominance* partial order $\trianglerighteq$ on compositions is

$$\alpha \trianglerighteq \beta \qquad \text{if } \textstyle\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \beta_i, \text{ for each } j \geq 1.$$

A λ-*tableau* is a bijection $t : [\lambda] \to \{1, \dots, n\}$. We use the notations:

- *tab*(λ) for the column standard λ-tableau: increasing down columns.
- *std*(λ) for the standard λ-tableau: increasing down columns and along rows.

A *partition of* n is a composition λ such that $\lambda_1 \geq \lambda_2 \geq \dots$. We write $\lambda \vdash n$. The transpose $[\lambda]'$ of the diagram of a partition λ is the diagram of the *transpose partition* λ' of n, defined by

$$\lambda_i' = \#\{j \mid \lambda_j \geq i\}, \quad \text{for each } i \geq 1.$$

A *tabloid* of n is an ordered set partition of $\{1, \dots, n\}$. We allow empty sets as parts of a tabloid. For example $(\{1, 2, 3\}, \{\}, \{4, 5\})$ and $(\{1, 2, 3\}, \{4, 5\}, \{\})$ are different tabloids of 5. Given a λ-tableau t, the entries in the rows of t form a tabloid, which we denote by $\{t\}$. We call this the *row-tabloid* of t. For example

$$\left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \right\} = \left\{ \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline 5 & 4 \\ \cline{1-2} \end{array} \right\} = (\{1, 2, 3\}, \{4, 5\})$$

If λ is a partition, the entries in the columns of t also form a tabloid, which we denote by $|t|$. We call this the *column-tabloid* of t. For example

$$\left|\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&\\\cline{1-2}\end{array}\right| = (\{1,4\},\{2,5\},\{3\}), \qquad \left|\begin{array}{|c|c|c|}\hline 2&1&3\\\hline 5&4&\\\cline{1-2}\end{array}\right| = (\{2,5\},\{1,4\},\{3\})$$

We define the *composition sequence* $(\tau^{(1)}, \tau^{(2)} \ldots, \tau^{(n)})$ of a column tabloid $|t|$ as follows:

$$\tau_j^{(i)} := \text{ the number of entries in column } j \text{ of } |t| \text{ that are } \leq i.$$

In particular $\tau^{(i)}$ is a composition of i, and $\tau^{(n)} = \lambda'$, if $\lambda \vdash n$. Clearly $|t|$ can be recovered from its composition sequence. So we may write $|t| \longleftrightarrow (\tau^{(i)})$. For example

$$\begin{array}{|c|c|c|c|}\hline 2&4&1&5\\\hline 3&6&&\\\cline{1-2}\end{array} \qquad \longleftrightarrow \qquad ([0010],[1010],[2010],[2110],[2111],[2211])$$

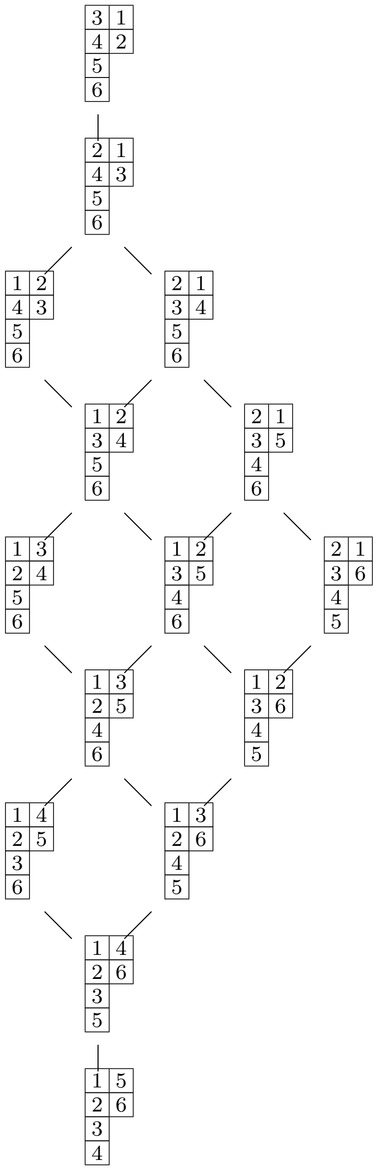
G. D. James implicitly uses the following total order $>$ on column-tabloids in order to prove that the standard polytabloids form a basis for a Specht module. If s, t are tableau, then

$|t| > |s|$ if for the largest i in which they differ, i occurs further to the left in t than in s.

However, such a total order is too crude for our purposes, as we will need to keep track of the positions of each of the entries $1, \ldots, \ell$ in a column-tabloid. For this reason, we use the following partial order $\trianglerighteq$ on column-tabloids. This is similar to the partial order on row-tabloids considered by James. Suppose that $|t|$ has composition sequence $(\tau^{(i)})$ and $|s|$ has composition sequence $(\sigma^{(i)})$. Then

$$|t| \trianglerighteq |s| \quad \text{if and only if} \quad \tau^{(i)} \trianglelefteq \sigma^{(i)} \text{ for each } i \geq 1.$$

For example, here is the $\trianglerighteq$ lattice for $(2^2, 1^2)$ column-tabloids:



Note that the most dominant λ column-tabloid is obtained by filling $[\lambda]$ with $n, \ldots, 2, 1$ by columns, working from left-to-right. Likewise, the least dominant λ column-tableau is obtained by filling $[\lambda]$ with $1, 2, \ldots, n$ by columns, again working from left-to-right.

We need the following elementary result:

Lemma 1. *Let $|t|$ be a column tabloid and let $X = \{x_1 < \cdots < x_r\}$ be a subset of column c of $|t|$ and $Y = \{y_1 < \cdots < y_r\}$ be a subset of column $c+1$ of $|t|$, with $x_1 > y_r$. Set $\pi = \prod_{i=1}^{r}(x_i, y_i)$ and suppose that $|t| \longleftrightarrow (\tau^{(i)})$ and $|t\pi| \longleftrightarrow (\sigma^{(i)})$. Then*

$$\tau^{(i)} \begin{cases} \lhd & \sigma^{(i)}, \qquad \textit{for } y_1 \leq i < x_r. \\ = & \sigma^{(i)}, \qquad \textit{for all other values of } i. \end{cases}$$

In particular, $|t| \rhd |t\pi|$.

Proof. It suffices to prove the result when $r = 1$. Now $\tau_j^{(i)} = \sigma_j^{(i)}$, unless $y_1 \leq i < x_1$ and $j \in \{c, c+1\}$. Moreover, if $y_1 \leq i < x_1$, then

$$\tau_c^{(i)} = \sigma_c^{(i)} - 1, \quad \tau_{c+1}^{(i)} = \sigma_{c+1}^{(i)} + 1.$$

It follows that $\tau^{(i)} \lhd \sigma^{(i)}$ for all such i. The lemma follows easily from these facts. □

2.2. **Specht modules and Garnir relations.** In this section λ is a partition of n and t is a λ-tableau. We use S_t to denote the row stabilizer of t. So S_t is conjugate in S_n to the Young subgroup S_λ and $\{t\} = \{t\pi \mid \pi \in S_t\}$. Set t' as the transpose of t. Then t' is a λ'-tableau. Clearly $S_{t'}$ coincides with the column stabilizer of t and $|t| = \{t\pi \mid \pi \in S_t'\}$.

For $H \leq S_n$, we have elements of RS_n defined by

$$H^+ := \sum_{\sigma \in H} \sigma, \qquad H^- = \sum_{\sigma \in H} \mathrm{sgn}(\sigma)\sigma.$$

As is customary, M^λ denotes the (right) RS_n-permutation module with basis the λ-tabloids. Here $\{t\}\pi := \{t\pi\}$, for each $\pi \in S_n$. Note that $M^\lambda \cong S_t^+ RS_n$, as right RS_n-modules.

The Specht module S^λ is the RS_n-submodule of M^λ generated by any one λ-polytabloid:

$$e_t := \{t\} S_{t'}^-.$$

Then $e_t\pi = e_{t\pi}$, for each $\pi \in S_n$ and in particular we have the *Column relation*

$$e_t\pi = \mathrm{sgn}(\pi) e_t, \quad \text{for each } \pi \in S_{t'}. \tag{1}$$

Also $S^\lambda \cong S_t^+ S_{t'}^- RS_n$, as RS_n-modules. It is known that $\{e_s \mid s \in \mathit{std}(\lambda)\}$ is an R-basis of S^λ. This is the *standard basis* of S^λ.

We call X, Y a *Garnir pair* for t if for some $c \geq 1$, X is a subset of column c of t, Y is a subset of column $c+1$ of t, and $|X| + |Y| > \lambda'_c$ (= the length of column c). Let $G_{X,Y}$ be a set of coset representatives (including 1) for the subgroup $S_X \times S_Y$ in $S_{X \cup Y}$. The *Garnir relation* corresponding to X, Y is

$$e_t G_{X,Y}^- = 0, \quad \text{or} \quad e_t = \sum_{1 \neq \sigma \in G_{X,Y}} -\mathrm{sgn}(\sigma)\, e_{t\sigma}. \tag{2}$$

Now suppose that t is column standard. We say that X, Y is an *elementary Garnir pair* for t if there exist nodes $(r, c), (r, c+1) \in [\lambda]$ such that X is the set of entries in or below row r in column c of t, Y is the set of entries in or above row r in column $c+1$ of t, and $x > y$ for all $x \in X$ and $y \in Y$. If $X = \{x_1 < \cdots < x_r\}$ and $Y = \{y_1 < \cdots < y_q\}$

then columns c and $c+1$ of t look like (c.f. [4, Section 8])

$$\begin{array}{ccc} * & & y_1 \\ & & \wedge \\ \vdots & & \vdots \\ & & \wedge \\ x_1 & > & y_q \\ \wedge & & \vdots \\ \vdots & & * \\ \wedge & & \\ x_r & & \end{array}$$

We have the following result relating to our partial order $\trianglerighteq$ on column tabloids.

Lemma 2. *Suppose that X, Y is an elementary Garnir pair for $t \in tab(\lambda)$. Then*

$$|t| \trianglerighteq |t\sigma|, \quad \textit{for each } \sigma \in G_{X,Y}.$$

Proof. We can factorize σ as the product of a permutation in C_t times $\prod_{i=1}^{r}(x_i, y_i)$ for $x_i \in X$ and $y_i \in Y$ as in Lemma 1. As the relation $\trianglerighteq$ depends only on the column sets of t and $t\sigma$, the result now follows from Lemma 1. □

Each polytabloid e_t can be expressed as a linear combination of standard polytabloids by applying a succession of elementary Garnir relations to e_t. This is the gist of the proof of Theorem 8.4 in [4]. We deduce that

> *The Specht module S^λ is the quotient of the free R-module generated by all polytabloids e_t by the RS_n-submodule generated by the column relations* (1) *and the elementary Garnir relations* (2).

Corollary 3. *Suppose that $t \in tab(\lambda)$. Then*

$$e_t = \sum_{s \in std(\lambda)} \zeta_s e_s,$$

where $\zeta_s \in R$, for all s, and $\zeta_s \neq 0$ implies that $|t| \trianglerighteq |s|$.

3. The Modules

3.1. RS_ℓ**-filtration on** S^λ**.** Fix a partition λ of n, a column-standard λ-tableau t and let $(\tau^{(i)})$ be the composition sequence of $|t|$. We use $t{\downarrow_I}$ to denote the restriction of t to an interval I of $\{1, \ldots, n\}$ for $t \in tab(\lambda)$. We call (λ, μ) a *partition pair* of (n, ℓ) and write

$$(\lambda, \mu) \vdash (n, \ell),$$

if $\mu \vdash \ell$ and $\mu_i \leq \lambda_i$, for each $i \geq 1$. So $[\mu] \subseteq [\lambda]$ and $[\lambda/\mu] := [\lambda] \backslash [\mu]$ is a skew diagram (with $n - \ell$ boxes). A λ/μ-tableau is a bijection $[\lambda/\mu] \to \{\ell+1, \ldots, n\}$. The dominance order on partition pairs is $(\lambda, \mu) \trianglerighteq (\alpha, \beta)$ if $\lambda \triangleright \alpha$, or $\lambda = \alpha$ and $\mu \trianglerighteq \beta$.

Fix $(\lambda, \mu) \vdash (n, \ell)$. Then we use the following notations:

- $tab(\lambda,\mu)=\{t\in tab(\lambda)\mid \tau^{(\ell)}=\mu'\}$.
- $tab(\lambda/\mu)$ the set of column standard skew λ/μ-tableau.
- $tab(\lambda,u)$, for $u\in tab(\lambda/\mu)$, the set of $t\in tab(\lambda,\mu)$ with $t{\downarrow_{[\ell+1,n]}}=u$.

Note that $t\in tab(\lambda,\mu)$ if and only if $t{\downarrow_{[1,\ell]}}\in tab(\mu)$ e.g. if $\lambda=(2^2,1^2)$ and $\mu=(2,1)$, then

$$
(3)\qquad
\begin{aligned}
tab(\lambda,\mu) \;&=\; \left\{
\begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline 5&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 2&1\\\hline 3&5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 2&1\\\hline 3&6\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&2\\\hline 3&6\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&3\\\hline 2&6\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\right\}\\
tab(\lambda/\mu) \;&=\; \left\{ u=
\begin{array}{|c|c|}\hline \ &\ \\\hline \ &4\\\hline 5&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},\;
v=
\begin{array}{|c|c|}\hline \ &\ \\\hline \ &5\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\;
w=
\begin{array}{|c|c|}\hline \ &\ \\\hline \ &6\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}
\right\}
\end{aligned}
$$

The first three tableau in $tab(\lambda,\mu)$ belong to $tab(\lambda,u)$, the second three to $tab(\lambda,v)$ and the last three to $tab(\lambda,w)$.

There is a bijection, preserving standard tableau

$$tab(\mu)\times tab(\lambda/\mu)\longleftrightarrow tab(\lambda,\mu).$$

Here $(s,u)\in tab(\mu)\times tab(\lambda/\mu)$ corresponds to $t\in tab(\lambda,\mu)$ if $s=t{\downarrow_{[1,\ell]}}$ and $u=t_{\downarrow_{[\ell+1,n]}}$. We indicate this by writing $t=s.u$.

We also need the following notations

- $tab(\lambda,\trianglelefteq\mu)=\{t\in tab(\lambda)\mid \tau^{(\ell)}\trianglerighteq\mu'\}$.
- $tab(\lambda,\triangleleft\mu)=\{t\in tab(\lambda)\mid \tau^{(\ell)}\triangleright\mu'\}$.

In any of the above definitions, we replace $tab(\,)$ by $std(\,)$ if the tableau are required to be standard. We note that in general

$$
(4)\qquad tab(\lambda,\trianglelefteq\mu)\;=\;tab(\lambda,\mu)\,\dot{\bigcup}\,tab(\lambda,\trianglelefteq\mu).
$$

Example: if $\lambda=(2^2,1^2)$ and $\mu=(2,1)$, then

$$
tab(\lambda,\triangleleft\mu)=\left\{
\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1} 6&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&4\\\hline 2&6\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1} 5&\multicolumn{1}{c}{}\\\cline{1-1}\end{array},
\begin{array}{|c|c|}\hline 1&5\\\hline 2&6\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1} 4&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}
\right\}
$$

Now consider the following R-subspaces of S^λ:

$$
\begin{aligned}
N^{\lambda/\mu} &:= sp\{e_t\mid t\in tab(\lambda,\trianglelefteq\mu)\},\\
J^{\lambda/\mu} &:= sp\{e_t\mid t\in tab(\lambda,\triangleleft\mu)\}.
\end{aligned}
$$

In particular $N^{\lambda/\nu}\subseteq N^{\lambda/\mu}$, for $(\lambda,\nu)\trianglelefteq(\lambda,\mu)$ and

$$J^{\lambda/\mu}=\sum_{\nu\triangleleft\mu}N^{\lambda/\nu}\quad\text{is contained in } N^{\lambda/\mu}.$$

Let $t \in tab(\lambda, \nu)$, with $\nu \trianglelefteq \mu$, and suppose that $\pi \in S_\ell$. Then $|t\pi| = |s|$, for some $s \in tab(\lambda, \nu)$. As $e_t\pi = e_{t\pi} = \pm e_s$, it follows that $N^{\lambda/\mu}$ and $J^{\lambda.\mu}$ are RS_ℓ-submodules of S^λ.

Lemma 4. *Let $t \in tab(\lambda, \trianglelefteq\mu)$ and $s \in tab(\lambda)$, with $t \trianglerighteq s$. Then $s \in tab(\lambda, \trianglelefteq\mu)$.*

Proof. This is an immediate consequence of the definitions. □

Corollary 5. *$N^{\lambda/\mu}$ has basis $\{e_t \mid t \in std(\lambda, \trianglelefteq\mu)\}$ and $J^{\lambda/\mu}$ has basis $\{e_t \mid t \in std(\lambda, \triangleleft\mu)\}$. Hence*

$$N^{\lambda/\mu}/J^{\lambda/\mu} \text{ has basis } \{e_t + J^{\lambda/\mu} \mid t \in std(\lambda/\mu)\}.$$

Proof. The last statement is a consequnce of the first two statements. These in turn are consequences of Corollary 3 and Lemma 4 and the fact that $\{e_t \mid t \in std(\lambda\}$ is a basis of S^λ. □

3.2. **An R-form for the $RS_n^{S_\ell}$-module** $\mathrm{Hom}_{RS_\ell}(S^\mu, S^\lambda)$**.** In this section (λ, μ) is a partition pair of (n, ℓ). For $u \in std(\lambda/\mu)$, let $(S^\mu)^u$ be an isomorphic copy of S^μ. For $s \in tab(\mu)$, denote the element of $(S^\mu)^u$ corresponding to $e_s \in S^\mu$ by e^u_s. Then

$$(S^\mu)^{\oplus std(\lambda/\mu)} := \bigoplus_{u \in std(\lambda/\mu)} (S^\mu)^u$$

is the external direct sum of $d(\lambda/\mu)$ copies of S^μ, one for each $u \in std(\lambda/\mu)$.

Theorem 6. *Set $\phi(e^u_s) := e_{s.u} + J^{\lambda/\mu}$, for each $(s, u) \in tab(\mu) \times std(\lambda/\mu)$. Then ϕ extends to an RS_ℓ-module isomorphism*

$$(S^\mu)^{\oplus std(\lambda/\mu)} \;\cong\; \frac{N^{\lambda/\mu}}{J^{\lambda/\mu}}.$$

Proof. First define ϕ only on the standard basis. So $\phi(e^u_s) = e_{s.u} + J^{\lambda/\mu}$, for each $(s, u) \in std(\mu) \times std(\lambda/\mu)$. Corollary 5 implies that ϕ extends to an R-isomorphism $(S^\mu)^{\oplus std(\lambda/\mu)} \longrightarrow N^{\lambda/\mu}/J^{\lambda/\mu}$.

Now let $(s, u) \in std(\mu) \times tab(\lambda/\mu)$, set $t := s.u \in tab(\lambda/\mu)$. Then $e_t\pi = e_{s\pi.u}$, for each $\pi \in S_\ell$. It follows that in order to show that ϕ is an RS_ℓ-homomorphism, and also that $\phi(e^u_s) := e_t + J^{\lambda/\mu}$, it is enough to show that the elements $e_t + J^{\lambda/\mu}$ of $N^{\lambda/\mu}/J^{\lambda/\mu}$ satisfy all column relations (1) and the standard Garnir relations of type (2) arising from s.

The column relations for e_s are satisfied. For, each column of s is contained in a column of t. So $S'_s \leq S'_t$ and (1) gives $e_t\pi = \mathrm{sgn}(\pi)e_t$, for each $\pi \in S'_s$. Suppose then that X, Y is a standard Garnir pair for s. Set $t := s.u \in tab(\lambda/\mu)$ and let Z be the set of entries $> \ell$ in the column occupied by X in t . Then $X \cup Z, Y$ is a standard Garnir pair for t and (2), applied in S^μ and then in S^λ, gives

$$e_sG^-_{X,Y} = 0, \quad \text{and} \quad e_tG^-_{X\cup Z,Y} = 0, \qquad \text{respectively.}$$

We claim that

$$(e_t + J^{\lambda/\mu})G^-_{X,Y} = 0.$$

This will follow if we show that

$$e_t\pi \in J^{\lambda/\mu}, \quad \text{for each } \pi \in G_{X\cup Z,Y}\backslash G_{X,Y}.$$

So let $\pi \in G_{X\cup Z,Y}\backslash G_{X,Y}$. After multiplication by an element of C_t, we may assume that $\pi = \pi' \prod_{i=1}^{r}(z_i, y_i)$, for some $\pi' \in S_\ell$, $z_i \in X$ and $y_i \in Y$. Now $t\pi' \in tab(\lambda,\mu)$. So Lemma 1 implies that $t\pi \in tab(\lambda, \lhd\mu)$, whence $e_t\pi \in J^{\lambda/\mu}$. This proves our claim, and completes the proof of the Theorem. □

Let λ_R denote the partition of $n-1$ obtained by removing a removable node R from $[\lambda]$. The *Branching rule* for Specht modules is

$$S_F^\lambda\!\downarrow_{S_{n-1}} = \sum_R S_F^{\lambda_R},$$

where R runs over the removable nodes of λ. The *Young graph* (or Bratelli diagram for $S_1 \leq \cdots \leq S_n \leq \ldots$) is formed by taking all partitions of the non-negative integers as nodes, and taking all pairs (λ, λ_R) as edges. There is a bijection between standard λ-tableau t defines a path in the Young graph that starts at λ and proceeds to the empty partition through $\lambda_{R_n}, \lambda_{R_nR_{n-1}}, \ldots$, where R_i is the node occupied by i in t. This bijection restricts to a bijection between standard λ/μ-tableau and paths in the branching graph starting at λ and ending at μ. We may iterate the branching rule to show that:

$$\dim\left(\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)\right) = \#std(\lambda/\mu). \tag{5}$$

Note that $\mathrm{Hom}_{RS_\ell}(S^\mu, S^\lambda)$ is a full R-lattice in $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$. In particular its R-rank is $\#std(\lambda/\mu)$.

Let δ be the most dominant μ-tableau: $[\mu]$ filled by rows from left to right and top to bottom.

Corollary 7. *Given $u \in std(\lambda/\mu)$, there exists an FS_ℓ-homomorphism $\theta_u : S_F^\mu \to S_F^\lambda$ such that*

$$\theta_u(e_\delta) = e_{\delta.u} + \epsilon_u, \qquad \textit{for some } \epsilon_u \in J^{\lambda/\mu}.$$

Moreover, $\{\theta_u \mid u \in std(\lambda/\mu)\}$ is a basis for $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$.

Proof. Theorem 6 enables us to construct a short exact sequence of FS_ℓ-modules

$$0 \longrightarrow J_F^{\lambda/\mu} \longrightarrow N_F^{\lambda/\mu} \xrightarrow{\phi} (S^\mu)_F^{\oplus std(\lambda/\mu)} \longrightarrow 0.$$

This sequence splits over FS_ℓ, as every FS_ℓ-module is semisimple. Let ϕ^{-1} be a map that splits ϕ. Then $\phi^{-1}(e_\delta^u) = e_{\delta.u} + \epsilon_u$, for some $\epsilon_u \in J^{\lambda/\mu}$. Restricting ϕ^{-1} to $(S_F^\mu)^u$, we deduce the existence of the FS_ℓ-maps θ_u.

As $\{\theta_u(e_\delta) \mid u \in std(\lambda/\mu)\}$ are linearly independent elements of S^λ, it follows that $\{\theta_u \mid u \in std(\lambda/\mu)\}$ are linearly independent elements of $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$. That the θ_u form a basis of $\mathrm{Hom}_{FS_\ell}(S_F^\mu, S_F^\lambda)$ now follows from (5). □

For example, if $\lambda = (2^2, 1^2)$ and $\mu = (2,1)$, then $\mathrm{Hom}_{FS_2}(S^{(2,1)}, S^{(2^2,1^2)})$ is 3-dimensional. Now $\delta = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ and c.f. (3), it can be shown that

$$\begin{aligned}
\theta_u(e_\delta) &= e_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1} 6\\\cline{1-1}\end{array}} - \frac{1}{3}e_{\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}} + \frac{1}{3}e_{\begin{array}{|c|c|}\hline 1&4\\\hline 2&6\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}}\\
\theta_v(e_\delta) &= e_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1} 6\\\cline{1-1}\end{array}} - \frac{1}{3}e_{\begin{array}{|c|c|}\hline 1&5\\\hline 2&4\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}} + \frac{1}{3}e_{\begin{array}{|c|c|}\hline 1&5\\\hline 2&6\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}}\\
\theta_w(e_\delta) &= e_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&6\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array}} - \frac{1}{3}e_{\begin{array}{|c|c|}\hline 1&4\\\hline 2&6\\\hline 3\\\cline{1-1} 5\\\cline{1-1}\end{array}} + \frac{1}{3}e_{\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}}
\end{aligned}$$

Of course, the whole point of this note is that we don't need those fractional terms in $J^{\lambda/\mu}$ in order to figure out the action of $RS_n^{S_\ell}$ on an R-form for $\mathrm{Hom}_{FS_2}(S^{(2,1)}, S^{(2^2,1^2)})$.

We now give a key definition. Consider the following elements of $N^{\lambda/\mu}/J^{\lambda/\mu}$:

$$e_u := e_{\delta.u} + J^{\lambda/\mu}, \quad \text{for each } u \in tab(\lambda/\mu).$$

Then the following is an R-subspace of $N^{\lambda/\mu}/J^{\lambda/\mu}$:

$$S^{\lambda/\mu} := sp\{e_u \mid u \in std(\lambda/\mu)\}.$$

Theorem 8. *$S^{\lambda/\mu}$ is an $RS_n^{S_\ell}$-submodule of $N^{\lambda/\mu}/J^{\lambda/\mu}$ and there is an $FS_n^{S_\ell}$-module isomorphism*

$$\begin{array}{rcll} \theta : S_F^{\lambda/\mu} & \longrightarrow & \mathrm{Hom}_{FS_\ell}(S^\mu, S^\lambda), & \textit{such that} \\ \theta(e_u) & = & \theta_u, & \textit{for each } u \in std(\lambda/\mu). \end{array}$$

Proof. It follows from Corollary 7 that

$$S^{\lambda/\mu} = \{\phi(e_\delta) \mid \phi \in \mathrm{Hom}_{RS_\ell}(S^\mu, N^{\lambda/\mu}/J^{\lambda/\mu})\}.$$

Suppose that $z \in RS_n^{S_\ell}$ and $\phi \in \mathrm{Hom}_{RS_\ell}(S^\mu, N^{\lambda/\mu}/J^{\lambda/\mu})$. Then $\phi(e_\delta)z = (\phi z)(e_\delta)$. As $\phi z \in \mathrm{Hom}_{RS_\ell}(S^\mu, N^{\lambda/\mu}/J^{\lambda/\mu})$, we deduce that $\phi(e_\delta)z \in S^{\lambda/\mu}$. So $S^{\lambda/\mu}$ is $RS_n^{S_\ell}$-invariant.

Comparison of dimensions show that there is an F-isomorphism $\theta : S_F^{\lambda/\mu} \longrightarrow \mathrm{Hom}_{FS_\ell}(S^\mu, S^\lambda)$, such that $\theta(\phi(e_\delta)) = \phi$. But θ is an FS_ℓ-isomorphism, as

$$\theta(\phi(e_\delta)z) = \theta((\phi z)(e_\delta)) = \phi z = \theta(\phi(e_\delta))z.$$

□

3.3. **Garnir relations for $S^{\lambda/\mu}$.** Let $v \in tab(\lambda/\mu)$ (column standard but not necessarily standard). In order to understand the action of $RS_n^{S_\ell}$ on $S^{\lambda/\mu}$, we need relations which allow us to write e_v in terms of the basis $\{e_u \mid u \in std(\lambda/\mu)\}$.

To this end, we call X, Y a *standard Garnir pair* for v if there exists $(r,c), (r,c+1) \in [\lambda/\mu]$ such that X consists of the entries in column c in or below row r in v, while Y consists of the entries in column $c+1$ in or above row r in v. Let $G_{X,Y}$ be a set of coset representatives (including 1) for the subgroup $S_X \times S_Y$ in $S_{X \cup Y}$.

Theorem 9. *[Elementary Garnir relations] Let $v \in tab(\lambda/\mu)$ and let X, Y be a standard Garnir pair for v. Then*

$$e_v G^-_{X,Y} = 0.$$

Proof. We consider the polytabloid $e_{\delta.v}$ in $N^{\lambda/\mu}$. Let Y_1 be the set of entries in column $c+1$ of δ. Then $X, Y_1 \cup Y$ is a standard Garnir pair for $e_{\delta.v}$. In particular

$$e_{\delta.v} G^-_{X,Y_1 \cup Y} = 0.$$

Now suppose that $\sigma \in G_{X,Y_1\cup Y} \backslash G_{X,Y}$. The effect of σ is to interchange some entries $> \ell$ in column c of $\delta.v$ with entries $< \ell$ in column $c+1$ of $\delta.v$. Let $\delta.v \leftrightarrow (\tau^{(i)})$ and $(\delta.v)\sigma \leftrightarrow (\rho^{(i)})$. Then it is clear that

$$\rho_i^{(\ell)} \begin{cases} > \tau_i^{(\ell)}, & \text{for } i = c. \\ < \tau_i^{(\ell)}, & \text{for } i = c+1. \\ = \tau_i^{(\ell)}, & \text{for } i \neq c, c+1. \end{cases}$$

So $\rho^{(\ell)} \rhd \tau^{(\ell)} = \mu'$, or $(\delta.v)\sigma \in tab(\lambda, \lhd\mu)$. It follows that $e_{\delta.v}\sigma \in J^{\lambda/\mu}$. The result follows easily from this. □

Corollary 10. $S^{\lambda/\mu} = sp\{e_v \mid v \in tab(\lambda/\mu)\}$*. All relations between the e_v arise from the column relations $e_v\pi = \operatorname{sgn}(\pi)e_v$, for $\pi \in S_{v'}$ and the elementary Garnir relations of the previous theorem.*

3.4. **Action of the Murphy elements on $S^{\lambda/\mu}$.** We consider the action of the Murphy elements $L_{\ell+1}, \ldots, L_n$ on $S^{\lambda/\mu}$. Suppose that $v \in std(\lambda/\mu)$ and $i \in \{\ell+1, \ldots, n\}$. Set $\alpha = c - r$, where i occupies node (r, c) in $[\lambda/\mu]$.

If C is the set of elements in a column of $\delta.v$ to the left of i in v, we have

$$\sum_{j \in C} e_{\delta.v}(i, j) = e_{\delta.v},$$

while if j and i share a column in $\delta.v$, we have

$$e_{\delta.v}(i, j) = -e_{\delta.v}.$$

Let

$$\begin{aligned} Y &= \{y \mid y < i \text{ and } y \text{ is to the right of } i \text{ in } v\}, \\ Y_{>\ell} &= Y \cap \{\ell+1, \ldots, n\}, \\ Z &= \{z \mid z > i \text{ and } z \text{ is to the left of } i \text{ in } v\}. \end{aligned}$$

Following Murphy [6], we obtain

$$e_{\delta.v} L_i = \alpha e_{\delta.v} - \sum_{z \in Z} e_{\delta.v}(i, z) + \sum_{y \in Y} e_{\delta.v}(i, y).$$

Then considering this equation in $N^{\lambda/\mu}/J^{\lambda/\mu}$, we obtain an identity in $S^{\lambda/\mu}$

$$e_v L_i = \alpha e_v - \sum_{z \in Z} e_{v(i,z)} + \sum_{y \in Y_{>\ell}} e_{v(i,y)}.$$

Of course, we have to apply the relations of Corollary 10 in order to express this in terms of the standard basis e_u, $u \in std(\lambda/\mu)$.

For example, we consider the action of $RS_6^{S_3}$ on $S^{(2^2,1^2)/(2,1)}$. The algebra is generated by $Z(RS_3)$, the transpositions $(4,5),(5,6)$ and the Murphy elements L_4, L_5, L_6. The elements of $Z(RS_3)$ acts as scalars on the Specht module $S^{(2,1)}$, and by the same scalars on $S^{(2^2,1^2)/(2,1)}$. Next we have, with the notation of (3), the following representing matrices with respect to the ordered basis e_u, e_v, e_w:

$$(4,5) \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (5,6) \longrightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L_4 \longrightarrow \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad L_5 \longrightarrow \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \end{bmatrix} \quad L_6 \longrightarrow \begin{bmatrix} -3 & 0 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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