PAPER II

1. If $z = x + iy$ ($x, y \in R$), express z^2 and $\frac{1+z}{1-z}$ in the form $a + ib(a, b \in R)$.
Find the value of x and the value of y for which
$$\frac{z+3i}{z+2} = \frac{2z+3i}{z+4}.$$

2. If $x \in R$, show that
$$-1 < x^3 - x \quad \text{when } 0 \leqslant x \leqslant 1$$
and that
$$x^3 - x > 0 \quad \text{when } x > 1.$$
Deduce that $x^3 - x + 1 = 0$ has no positive root.
If α, β, γ are the roots of the equation $x^3 - x + 1 = 0$, find the equations whose roots are
(i) $-\alpha, -\beta, -\gamma$, (ii) $\frac{\alpha}{10}, \frac{\beta}{10}, \frac{\gamma}{10}$.

3. (a) State which of the following statements are true and which are false (A and B are sets):
(i) $x \in A$ and $x \in B \Rightarrow x \in A \cup B$
(ii) $x \notin A$ or $x \notin B \Rightarrow x \notin A \cap B$
(iii) $x \in A \cap B \Rightarrow x \in A \cup B$
(iv) $x \notin A$ and $x \notin B \Rightarrow x \notin A \cap B$
[The symbol $\Rightarrow$ denotes "implies"].

(b) If E and F are subsets of a universal set U and $E \dot{\cup} F = F$, simplify each of the following:
(i) $E \cap F$, (ii) $E' \cup F$, (iii) $E \cap F'$.

(c) If P and Q are non-empty subsets of T and $P \cap Q = P$, find the set $X \subset T$ which simultaneously satisfies the two equations
$$X \cup P = Q,$$
$$X \cap P = \phi.$$

4. (a) If $0 < x < 1$ ($x \in R$), show that $\frac{1}{x} - 1 > 0$.
If $y = \frac{1}{x} - 1$, show that $x = \frac{1}{1+y}$ and that $x^n < \frac{1}{ny}$ ($n \in N$, $n > 0$).
If $e > 0$, find $k \in R$ in terms of e and y for which $\frac{1}{ny} < e$ for all $n > k$.
Deduce that $\lim_{n \to \infty} x^n = 0$.
Show also that $nx^n < \frac{2}{(n-1)y^2}$ and hence prove that the sequence
$$x, 2x^2, 3x^3, \dots nx^n, \dots$$
is convergent.

(b) Prove that the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges if $0 < x < 1$.

5. (a) Differentiate from first principles $\frac{1}{x^3}$ with respect to x.
Differentiate with respect to x:
(i) $\frac{x^3 - 1}{x^3}$ (ii) $x\sin(x^3 - 5)$ (iii) $e^{\left(\frac{x}{\tan x}\right)}$.

(b) If $y = (A + Bx)e^{-2x}$ (A and B independent of x), prove that

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

If $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$, find the value of A and the value of B.

6. A piece of wire 20 cm. in length is cut into two pieces. One piece is bent to form a square and the other piece is bent to form a circle. If the sum of the area of the square and the area of the circle is a minimum, prove that the length of the side of the square is equal to the length of the diameter of the circle.

7. (a) Evaluate

$$\text{(i)} \int_{-1}^{1}(1 + x)^3 dx, \qquad \text{(ii)} \int_{0}^{\frac{\pi}{2}} \sin^2 2x \cos x \, dx, \qquad \text{(iii)} \int_{0}^{1} xe^{1 - x^2} dx.$$

(b) The line segment $hy = rx (0 \leq x \leq h)$, h and r are constants) is rotated about the x-axis so as to generate a cone. Show that the volume of the cone is $\frac{1}{3}\pi r^2 h$.

8. (a) A die is thrown three times. What is the probability of obtaining
   (i) a six each time
   (ii) a six on the third throw only ?

(b) A person buys a certain number of tickets so that the probability he wins a prize is 1%
Assuming that the binomial distribution applies, find the least number of tickets that must be drawn if the probability of his getting a prize is greater than 80%.

9. If $y^2 = x(x - 3)(x - 8)$, find the domain of values of x for which $y \in R$.
Show that the graph of $y^2 = x(x - 3)(x - 8)$ is symmetrical about the x-axis and find three points of the graph at which the tangents to the graph are parallel to the y-axis.
Trace the graph, paying special attention to the maximum and minimum points and to the shape of the graph as x tends to infinity.

10. Plot the set of couples (ordered pairs) (x, y) which simultaneously satisfy the inequalities

$$x \geq 0, \; y \geq 0, \; x + 2y \leq 40, \; 2x + y \leq 40, \; 8x + 8y \leq 200.$$

Find the coordinates of each vertex of the set.
A factory manufactures two items A and B. Three machines M_1, M_2, M_3 are used to manufacture each item and the number of hours spent by each machine on each item is given in the following table:

	M_1	M_2	M_3
A	1	2	$1\frac{3}{5}$
B	2	1	$1\frac{3}{5}$

No machine can work more than 40 hours per week. If the profit on each item A is £15 and on each item B is £20, find the number of each item which should be manufactured per week so as to maximise the profit, assuming that all items made are sold.