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LEAVING CERTIFICATE EXAMINATION, 2000

MATHEMATICS — HIGHER LEVEL — PAPER 1 (300 marks)

THURSDAY, 8 JUNE — MORNING, 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

Marks may be lost if all necessary work is not clearly shown.

1. **(a)** Show that the following simplifies to a constant when $x \neq 2$

$$\frac{3x-5}{x-2}+\frac{1}{2-x}.$$

(b) $f(x) = ax^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbf{R}$.

If k is a real number such that $f(k) = 0$, prove that $x - k$ is a factor of $f(x)$.

(c) $(x-t)^2$ is a factor of $x^3 + 3px + c$.

Show that

(i) $p = -t^2$

(ii) $c = 2t^3$.

2. **(a)** Solve for x, y, z

$$3x - y + 3z = 1$$
$$x + 2y - 2z = -1$$
$$4x - y + 5z = 4.$$

(b) Solve $x^2 - 2x - 24 = 0$.

Hence, find the values of x for which

$$\left(x+\frac{4}{x}\right)^2 - 2\left(x+\frac{4}{x}\right) - 24 = 0, \qquad x \in \mathbf{R},\ x \neq 0.$$

(c) **(i)** Express $a^4 - b^4$ as a product of three factors.

(ii) Factorise $a^5 - a^4b - ab^4 + b^5$.

Use your results from **(i)** and **(ii)** to show that

$$a^5 + b^5 > a^4b + ab^4$$

where a and b are positive unequal real numbers.

3. **(a)** Given that $A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ -5 & -2 \end{pmatrix}$, find $B^{-1}A$.

(b) **(i)** Simplify $\left(\dfrac{-2+3i}{3+2i}\right)$ and hence, find the value of $\left(\dfrac{-2+3i}{3+2i}\right)^9$ where $i^2 = -1$.

(ii) Find the two complex numbers $a + ib$ such that

$$(a+ib)^2 = 15 - 8i.$$

(c) Use De Moivre's theorem

(i) to prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(ii) to express $\left(-\sqrt{3} - i\right)^{10}$ in the form $2^n\left(1 - i\sqrt{k}\right)$ where $n, k \in \mathbf{N}$.

4. **(a)** The first three terms of a geometric sequence are

$$2x - 4, \quad x + 1, \quad x - 3.$$

Find the two possible values of x.

(b) Given that

$$u_n = \frac{1}{2}\left(4^n - 2^n\right)$$

for all integers n, show that

$$u_{n+1} = 2u_n + 4^n.$$

(c) **(i)** Given that $g(x) = 1 + 2x + 3x^2 + 4x^3 \ldots$ where $-1 < x < 1$, show that

$$g(x) = \frac{1}{(1-x)^2}.$$

(ii) $P(n) = u_1 u_2 u_3 u_4 \ldots u_n$ where

$u_k = ar^{k-1}$ for $k = 1, 2, 3, \ldots, n$ and $a, r \in \mathbf{R}$.

Write $P(n)$ in the form $a^n r^{f(n)}$ where $f(n)$ is a quadratic expression in n.

5. **(a)** Express the recurring decimal $1.\dot{2}$ in the form $\frac{a}{b}$ where $a, b \in \mathbf{N}$.

(b) Prove by induction that $n! > 2^n$, $n \in \mathbf{N}$, $n \geq 4$.

(c) **(i)** Solve for x

$$2\log_9 x = \frac{1}{2} + \log_9(5x + 18), \quad x > 0.$$

(ii) Solve for x

$$3e^x - 7 + 2e^{-x} = 0.$$

6. **(a)** Differentiate with respect to x

(i) $(1 + 5x)^3$

(ii) $\frac{7x}{x - 3}, \quad x \neq 3.$

(b) **(i)** Prove, from first principles, the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $u = u(x)$ and $v = v(x)$.

(ii) Given $y = \sin^{-1}(2x - 1)$, find $\frac{dy}{dx}$ and calculate its value at $x = \frac{1}{2}$.

(c) $f(x) = \frac{1}{x + 1}$ where $x \in \mathbf{R}$, $x \neq -1$.

(i) Find the equations of the asymptotes of the graph of $f(x)$.

(ii) Prove that the graph of $f(x)$ has no turning points or points of inflection.

(iii) If the tangents to the curve at $x = x_1$ and $x = x_2$ are parallel and if $x_1 \neq x_2$, show that

$$x_1 + x_2 + 2 = 0.$$

7. **(a)** Find the slope of the tangent to the curve

$$x^2 - xy + y^2 = 1 \text{ at the point } (1,0).$$

(b) The parametric equations of a curve are

$$x = \cos^3 t \text{ and } y = \sin^3 t, \quad 0 \le t \le \frac{\pi}{2}.$$

(i) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in terms of t.

(ii) Hence, find integers a and b such that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a}{b}(\sin 2t)^2.$$

(c) $f(x) = \frac{\ln x}{x}$ where $x > 0$.

(i) Show that the maximum of $f(x)$ occurs at the point $\left(e, \frac{1}{e}\right)$.

(ii) Hence, show that $x^e \le e^x$ for all $x > 0$.

8. **(a)** Find **(i)** $\int (x^2 + 2)dx$ **(ii)** $\int e^{3x} dx$.

(b) Evaluate **(i)** $\int_0^{\frac{\pi}{2}} \sin^2 3\theta \, d\theta$ **(ii)** $\int_0^1 \frac{x}{x^2 + 4} dx$.

(c) **(i)** Find the value of the real number p given that

$$\int_2^p \frac{dx}{x^2 - 4x + 5} = \frac{\pi}{4}.$$

(ii) The region bounded by the curve $y = x^2$ and the line $y = 4$ is divided into two regions of equal area by the line $y = k$.

Show that $k^3 = 16$.