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LEAVING CERTIFICATE EXAMINATION, 1999 25425

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 10 JUNE - MORNING 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each).

Marks may be lost if necessary work is not clearly shown or you do not indicate where a calculator has been used.

1. **(a)** Show that $\frac{-1+\sqrt{3}}{1+\sqrt{3}} = 2-\sqrt{3}$.

(b) Solve for x

$$\frac{4x-1}{x-3} < 2, \quad x \in \mathbf{R} \text{ and } x \neq 3.$$

(c) $x^2 + bx + c$ is a factor of $x^3 - p$.

Show that

(i) $b^3 = p$

(ii) $c^3 = p^2$.

2. **(a)** Solve the simultaneous equations

$$x + y = 1$$
$$x^2 + y^2 = 25.$$

(b) If for all integers n,

$$u_n = 2^{2n-1} + 2^{n-1} ,$$

show that

$$u_{n+1} - 2u_n - 2^{2n} = 0.$$

(c) Let a, b, c be positive unequal real numbers.

Using the results $a^2 + b^2 > 2ab$, $b^2 + c^2 > 2bc$ and $c^2 + a^2 > 2ac$,

(i) deduce that $a^2 - ab + b^2 > ab$

(ii) deduce that $a^2 + b^2 + c^2 > bc + ca + ab$

(iii) show that $a^3 + b^3 > ab(a + b)$.

3. **(a)** If $A = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}$, find A^{-1}.

(b) **(i)** Find a quadratic equation whose roots are $3 + i$ and $3 - i$, where $i^2 = -1$.

(ii) Let $P(z) = z^3 - kz^2 + 22z - 20,\ k \in \mathbf{R}$.

$3 + i$ is a root of the equation $P(z) = 0$.
Find the value of k.
Find the other two roots of the equation $P(z) = 0$.

(c) **(i)** Solve for w

$$\sqrt{5}|w| + iw = 3 + i\ .$$

Write your answers in the form $u + iv,\ \ u, v \in \mathbf{R}$.

(ii) Use De Moivre's theorem to find three roots of the equation

$$z^6 - 1 = 0\ .$$

4. **(a)** Solve $\binom{n+4}{2} = 91$, for $n \in \mathbf{N}$.

(b) **(i)** The nth term of an arithmetic series is $3n + 2$.
Find S_n, the sum of the first n terms, in terms of n.

(ii) Evaluate, in terms of n, $\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right)$.

(c) Let $f(x) = \sum_{n=1}^{\infty} q^{n-1}x^n$, where $|x| < 1$ and $0 < q < 1$.

Show that $f(x) = \dfrac{x}{1-qx}$.

If $g(x) = \dfrac{1}{1-(1-q)f(x)}$, show that $g(x) = \dfrac{1-qx}{1-x}$.

5. (a) Find the coefficient of a^3 in the expansion of $(2+a)^5$.

(b) (i) Solve the equation

$$\sqrt{2x+7} = 2+\sqrt{x}\ .$$

(ii) If $x > 0$ and $x \neq 1$, show that

$$\frac{1}{\log_2 x}+\frac{1}{\log_3 x}+\frac{1}{\log_5 x}=\frac{1}{\log_{30} x}\ .$$

Note: $\log_b a = \dfrac{\log_c a}{\log_c b}$.

(c) Prove by induction that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.

6. (a) Differentiate

$$(3-4x)^5$$

with respect to x.

(b) Find from first principles the derivative of $\sin x$ with respect to x.

(c) Let $f(x) = xe^{-ax}$, $x \in \mathbf{R}$, a constant and $a > 0$.

Show that $f(x)$ has a local maximum and express the coordinates of this local maximum point in terms of a.

Find, in terms of a, the coordinates of the point at which the second derivative of $f(x)$ is zero.

7. (a) Find the derivative of $\sqrt{x^2+1}$.

(b) (i) Let $x = t - \sin t \cos t$ and $y = 4\cos t$, $0 < t < \frac{\pi}{2}$.

Show that $\frac{dy}{dx} = -\frac{2}{\sin t}$.

(ii) Find the slope of the tangent to the curve

$$x^2 - y^2 - x = 1$$

at the point (2, 1).

(c) Let $f(x) = x^3 + kx^2 - 4$, $x \in \mathbf{R}$ and $k > 0$.

Show that the coordinates of the local minimum and local maximum of $f(x)$ are $(0, -4)$ and $\left(\frac{-2k}{3}, \frac{4k^3-108}{27}\right)$, respectively.

Find,

(i) the range of values of k for which $f(x) = 0$ has three real roots

(ii) the value of k for which $f(x) = 0$ has three roots, two of which are equal.

8. (a) Find $\int\left(4x+1+\frac{1}{x^3}\right)dx$.

(b) Evaluate (i) $\int_0^{\pi/6} 2\cos 4\theta \cos 2\theta \, d\theta$ (ii) $\int_{-3}^{0} (x+3)e^{x(x+6)}dx$.

(c) Evaluate $\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$.

Hint: let $x = 2\sin\theta$.