24523

AN ROINN OIDEACHAIS — M. 29

LEAVING CERTIFICATE EXAMINATION, 1997

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 12 JUNE – MORNING, 9·30 to 12·00

Attempt **SIX QUESTIONS** (50 marks each)

Marks may be lost if all your work is not clearly shown or you do not indicate where a calculator has been used.

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1. *(a)* If $x = \sqrt{a} + \dfrac{1}{\sqrt{a}}$ and $y = \sqrt{a} - \dfrac{1}{\sqrt{a}}$, $a > 0$, find the value of $\sqrt{x^2 - y^2}$.

(b) Let $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbf{R}$. If k is a real number such that $f(k) = 0$, prove that $x - k$ is a factor of $f(x)$.

(c) If $(x-1)^2$ is a factor of $ax^3 + bx^2 + 1$, find the value of a and the value of b.

2. *(a)* Solve the simultaneous equations

$$2x - 3y = 1$$
$$x^2 + xy - 4y^2 = 2.$$

(b) Solve

$$x^2 - 6x + 8 = 0$$

and hence find the values of x for which

$$\left(x + \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 8 = 0, \quad x \in \mathbf{R} \text{ and } x \neq 0.$$

(c) Let $f(x) = \dfrac{1}{x}$ for all $x \in \mathbf{R}$ and $x \neq 0$.

Points a and b have coordinates $(p, f(p))$ and $(q, f(q))$, respectively, for $0 < p < q$.

(i) Show that the equation of the line ab can be written as

$$y = g(x) = \frac{1}{p} - \frac{1}{pq}(x - p).$$

(ii) Show that

$$f(x) - g(x) = \frac{(x-q)(x-p)}{pqx}.$$

Hence, show that $f(x) - g(x) < 0$ for $0 < p < x < q$.

OVER→

3. *(a)* If $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$, find the matrix C such that $C = A(A - B)$.

(b) Let $P(z) = z^3 - (10 + i)z^2 + (29 + 10i)z - 29i$, where $i^2 = -1$.

(i) Determine the real numbers a and b if

$$P(z) = (z - i)(z^2 + az + b).$$

(ii) Plot on an Argand diagram the solution set of $P(z) = 0$.

(c) *(i)* Let $w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w_2 = (w_1)^2$.

Verify that

$$x^2 + xy + y^2 = (x - w_1y)(x - w_2y), \text{ where } x, y \in \mathbf{R}.$$

(ii) Express $2(1 - i\sqrt{3})$ in the form $r(\cos\theta + i\sin\theta)$.

Using De Moivre's theorem find values for

$$[2(1 - i\sqrt{3})]^{3/2}$$

and write your answers in the form $p + qi$, $p, q \in \mathbf{R}$.

4. *(a)* Write down, or find, in terms of n, the sum of n terms of the finite arithmetic series

$$1 + 2 + 3 + \ldots + n.$$

(b) If for all integers n,

$$u_n = (5n - 3)2^n,$$

verify that

$$u_{n+1} - 2u_n = 5(2^{n+1}).$$

(c) Consider the sum to n terms, S_n, of the following finite geometric series

$$S_n = 1 + (1 + x) + (1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^{n-1}$$

for $x > 0$.

Show that the coefficient of x^2 in the above expression for S_n is

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n-1}{2}.$$

By finding S_{25} in terms of x and by considering the coefficient of x^2 in S_{25}, find the value of p and the value of q for which

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{24}{2} = \binom{p}{q}, \text{ where } p, q \in \mathbf{N}.$$

5. *(a)* Solve

$$\log_5 x = 1 + \log_5\left(\frac{3}{2x-1}\right), \quad x \in \mathbf{R},\ x > \frac{1}{2}.$$

(b) *(i)* Solve $\dfrac{x+3}{x-4} < -2, \ x \neq 4, x \in \mathbf{R}.$

(ii) If k is a positive integer and 720 is the coefficient of x^3 in the binomial expansion of $(k + 2x)^5$, find the value of k.

(c) Prove by induction that 8 is a factor of $3^{2n} - 1$ for $n \in \mathbf{N}_0$.

6. *(a)* Differentiate

(i) $x^3 + 2\sqrt{x}$ *(ii)* $(x + 2)\ln x$.

(b) *(i)* Find from first principles the derivative of x^3 with respect to x.

(ii) Let $f(x) = \sin^4 x + \cos^4 x$.

Find the derivative of $f(x)$ and express it in the form $k\sin px$, where $k, p \in \mathbf{Z}$.

(c) If $\sin y = \frac{1}{2}(1 - x^2)$ for $-\sqrt{3} < x < \sqrt{3}$,

calculate the value of a and the value of b when

$$\left(\frac{dy}{dx}\right)^2 = \frac{a}{3 - x^2} - \frac{b}{1 + x^2}, \quad a, b \in \mathbf{N}_0.$$

OVER→

7. *(a)* Take $x_1 = 3$ as the first approximation of a real root of the equation

$$x^3 - 6x^2 + 24 = 0.$$

Find, using the Newton-Raphson method, x_2, the second approximation and write your answer as a fraction.

(b) *(i)* Find the equation of the tangent to the curve

$$2x^2 - 3y^2 = 6$$

at the point $(-3, -2)$.

(ii) If $x = \dfrac{1 - t^2}{1 + t^2}$ and $y = \dfrac{2t}{1 + t^2}$, find, as a fraction, the value of $\dfrac{dy}{dx}$ when $t = \dfrac{3}{4}$.

(c) Let $y = x - 1 + \dfrac{1}{x - 1}$, $x \in \mathbf{R}$, $x \neq 1$.

(i) Find the values of x for which $\dfrac{dy}{dx} = 0$.

(ii) For x real, show that y cannot have a real value between -2 and $+2$.

8. *(a)* Find *(i)* $\displaystyle\int \sin 4x\,dx$ *(ii)* $\displaystyle\int (1 + \sqrt{x})^2 dx$.

(b) Evaluate *(i)* $\displaystyle\int_{-\pi/2}^{\pi/2} 2\cos^2 3\theta\, d\theta$ *(ii)* $\displaystyle\int_0^1 \frac{x^2}{x + 1}\,dx$.

(c) Calculate the value of

$$\int_{1/3}^{3} \frac{1}{t + \sqrt{t}}\,dt .$$

Hint: let $u = \sqrt{t}$.