AN ROINN OIDEACHAIS

LEAVING CERTIFICATE EXAMINATION, 1996

MATHEMATICS — HIGHER LEVEL 22476
PAPER 2 (300 marks)

FRIDAY, 7 JUNE — MORNING, 9.30 to 12.00

Attempt **five** questions from Section **A** and **one** question from Section **B.** Each question carries 50 marks.

Marks may be lost if necessary work is not shown or if you do not indicate where a calculator has been used.

SECTION A

1. **(a)** The parametric equations of a circle are

$$x = 5 + \frac{\sqrt{3}}{2}\cos\theta, \quad y = -3 + \frac{\sqrt{3}}{2}\sin\theta.$$

Find its Cartesian equation.

(b) Points (1, –1), (–6, –2) and (3, –5) are on a circle C.

Find the equation of C.

(c) S_1: $x^2 + y^2 - 6x - 4y + 12 = 0$

S_2: $x^2 + y^2 + 10x + 4y + 20 = 0$ are two circles.

(i) Find the coordinates of their centres p and q and the lengths of their radii r_1, r_2 respectively.

(ii) Verify that the lines

$$L\colon y - 1 = 0 \quad \text{and} \quad M\colon 4x - 3y - 1 = 0$$

are tangents to S_1.

(iii) If w is the point of intersection of L and M and $w \in [pq]$, show that

$$|pw| : |wq| = r_1 : r_2.$$

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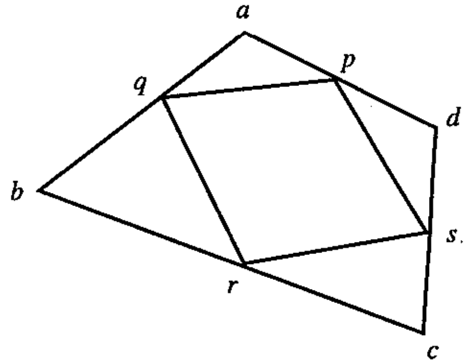
2. (a) $\vec{r} = 7\vec{i} - 4\vec{j}$ and $\vec{r}^{\perp}$ is its related vector $4\vec{i} + 7\vec{j}$.

$m\vec{r} + n\vec{r}^{\perp} = 5\vec{i} - 40\vec{j}$. Find the value of m and the value of n where m and n ε R.

(b) p, q, r, s are the mid-points of the sides of a quadrilateral $abcd$.

Prove by vector methods that $pqrs$ is a parallelogram.



(c) o is the origin, $\vec{a} = 2\vec{i} + 2\vec{j}$, $\vec{b} = 4\vec{i} + 4\vec{j}$.

If $\vec{r} = \frac{1}{2}\left(\vec{a} + \vec{b}\right) + t(\vec{b} - \vec{a})^{\perp}$, $t \ \varepsilon \ R$, express $\vec{r}$ in terms of $\vec{i}$, $\vec{j}$ and t.

Show that r lies on the perpendicular bisector of $[ab]$ for all $t \ \varepsilon$ R i.e. show that $|\overrightarrow{ra}| = |\overrightarrow{rb}|$.

3. (a) (i) The parametric equations of the lines L and K are:

$$L: x = t + \frac{1}{2}, \quad y = 2t + 7$$

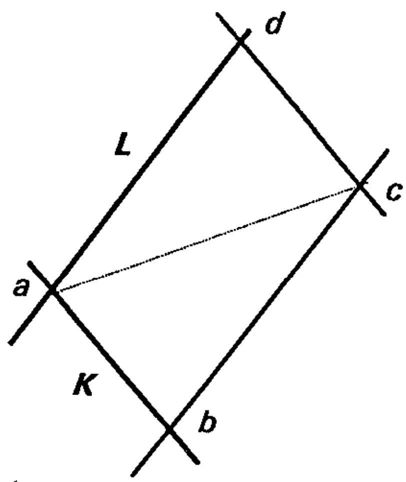
$$K: x = \frac{1-t}{3}, \quad y = t - 5.$$

Show that their Cartesian equations are

$$L: 2x - y + 6 = 0$$
$$K: 3x + y + 4 = 0$$

and find a, their point of intersection.

(ii) If L and K contain adjacent sides of a parallelogram $abcd$ and the mid-point of $[ac]$ is $\left(0, 3\frac{1}{2}\right)$, find the coordinates of vertices c, b and d.



(b) If $p = (0,0)$ and $q = (1,2)$ show that

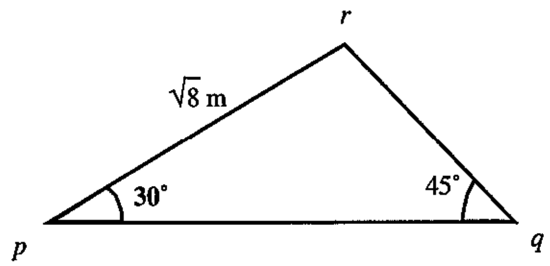
$$x = t, \ y = 2t, \quad 0 \leq t \leq 1$$

are parametric equations of the line segment $[pq]$.

Find the image of this segment under the transformation

$$x' = 3x - y$$
$$y' = x - y.$$

4. (a) Show that the area of the triangle pqr, correct to one decimal place, is 2.7 m^2, if $|pr| = \sqrt{8}$ m, $|\angle rpq| = 30°$ and $|\angle pqr| = 45°$.



(b) If $\tan(45° - A) = \dfrac{1 - \tan A}{1 + \tan A}$ show that $\dfrac{\cos 2A}{1 + \sin 2A} = \tan(45° - A)$.

[Note: $\sin^2 A + \cos^2 A = 1$.]

Deduce that $\tan 22\frac{1}{2}° = \dfrac{1}{\sqrt{2} + 1}$.

(c) [sp], [tq] are vertical poles each of height 10 m, p, q, r are points on level ground. Two wires of equal length join s and t to r, i.e. $|sr| = |tr|$.

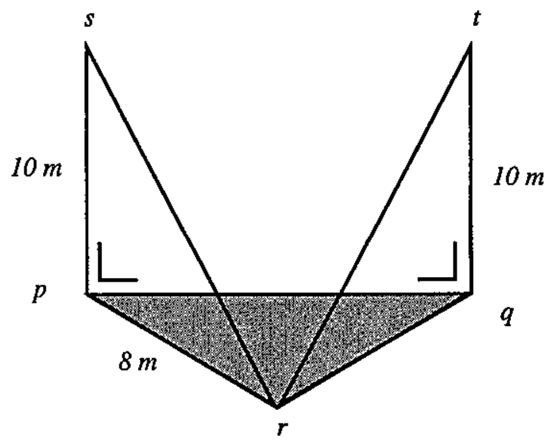
If $|pr| = 8$ m, $|\angle pqr| = 32°12'$, $|\angle prq| = 120°$,

calculate

(i) $|pq|$ to the nearest metre

(ii) $|sr|$ in surd form

(iii) $|\angle srt|$ to the nearest degree.



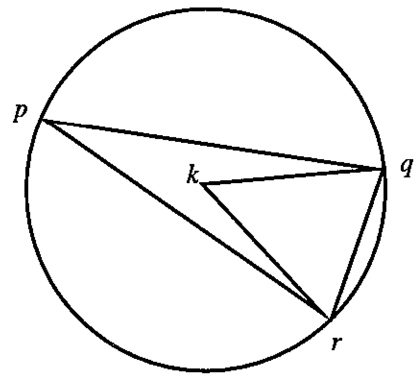
5. (a) Find the value k, if

$$k = \frac{\cos\left(\frac{\pi}{4} + \theta\right) - \cos\left(\frac{\pi}{4} - \theta\right)}{\sin\left(\frac{\pi}{4} + \theta\right) - \sin\left(\frac{\pi}{4} - \theta\right)} \quad \text{where } \sin\theta \neq 0.$$

(b) p,q,r are points of a circle, centre k. The length of the radius of the circle is 2 cm. The length of the minor arc pq is $\frac{5\pi}{3}$ cm.

(i) Find the length of the chord [pq], correct to two places of decimals.

(ii) If $|pq| = |pr|$, find $|rq|$.



(c) $x = 0°$ and $x = 60°$ are two solutions of the equation $a\sin^2 2x + \cos 2x - b = 0$ where $a,b \in N$.

Find the value of a and the value of b.

Using these values of a and b, find all the solutions of the equation where $0° \leq x \leq 360°$.

6. (a) In how many ways can a group of five be selected from ten people?

How many groups can be selected if two particular people from the ten cannot be in the same group?

(b) There are seven white and four black beads in a bag. A bead is picked at random and not replaced. A second bead is then picked.

(i) Find the probability that both beads are the same colour.

The two beads are returned to the bag and three red beads are added. Three beads are then picked at random without replacement. Find the probability that

(i) all three beads are different in colour

(ii) at least two beads of the same colour are picked.

(c) Show that

$$u_n = \frac{1}{3}\left\{(1 + \sqrt{3})^n - (1 - \sqrt{3})^n\right\}$$

is the solution of the difference equation

$$u_{n+2} - 2u_{n+1} - 2u_n = 0, \quad n \geq 0$$

when $u_0 = 0$ and $u_1 = \frac{2\sqrt{3}}{3}$.

Verify this solution.

7. (a) Four numbers have a mean p.

Five numbers have a mean x.

These nine numbers have a mean q.

Express x in terms of p and q.

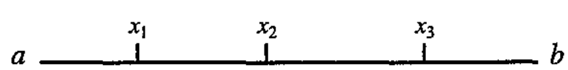
(b) Two dice A and B are cast. What is the probability of getting

(i) a total of two or a total of six?

(ii) a total greater than nine or a total which is prime?

(iii) a total which is three times as great as other possible totals?

(c) Real numbers x_1, x_2 and x_3 are each greater than a and less than b as shown on the number line.



Prove that

(i) $a < \bar{x} < b$ where $\bar{x}$ is the mean of x_1, x_2 and x_3.

(ii) $\sigma \leq b - a$ where σ is the standard deviation of x_1, x_2 and x_3.

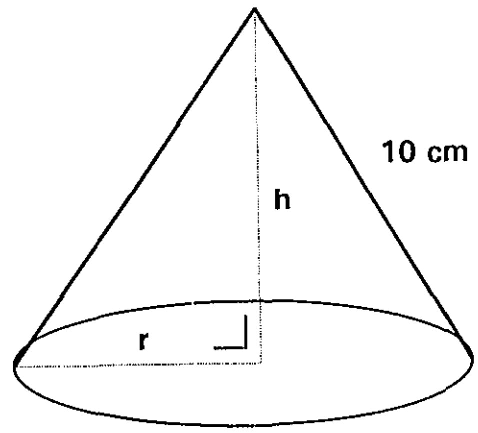
SECTION B

Answer ONE question from this section.

8. **(a)** Use integration by parts to find $\int xe^{-x}\,dx$.

(b) The slant length of a right circular cone is 10 cm, see diagram.

Find the maximum volume of the cone, in terms of π.



(c) Find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ for

$$f(x) = (1 + x)^m.$$

Hence write the first four terms and the $(r + 1)$th term of the Maclaurin series for $f(x) = (1 + x)^m$.

Test the series for convergence when $m \in Q \backslash N$.

9. **(a)** Three cards are drawn, one after the other, without replacement, from a pack of 52 playing cards.

Find the probability that the first is a king, the second is an ace, and the third neither an ace nor a king.

(b) In a game of chess against a particular opponent, the probability that Sean wins is $\frac{3}{5}$. He plays six games against this opponent. What is the probability that Sean will

(i) lose the second and fourth game and win the others?

(ii) win exactly four games?

(iii) lose at least four games?

(c) A company installs a new machine in a factory. The company claims that the machine will fill bags with sugar having a mean mass of 500 g and a standard deviation of 18 g.

36 bags are checked in a random sample. Their mean mass is 505 g. At the 5% level of significance is this result consistent with the company's claim?

10. (a) Verify that $\{0,1,2,3\}$ mod 4 is a group under the operation of addition. You may assume associativity under addition in N.

(b) G is the set of permutations of $\{1,2,3\}$. They are six in number, labelled

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$G,\circ$ is a group under composition of permutations where $\circ$ denotes composition.

(i) Find the inverse of d.

Write down the elements of $G,\circ$ of order two.

(ii) Show $f \circ g \neq g \circ f$.

(iii) Find the centraliser $C(f)$, of f, i.e. the set of elements that commute with f.

Show that $C(f)$ is a cyclic subgroup of $G,\circ$.

(c) Prove that in any group G, if $g \varepsilon G$, then the set H

$$H = \{g^n : n \varepsilon Z\}$$

is a subgroup.

11. (a) (i) Let f be the transformation $(x,y) \longrightarrow (x',y')$ where

$x' = ax$

$y' = by, \qquad a \neq b.$

If C is the circle $x^2 + y^2 = 1$, show that $f(C)$ is the ellipse

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

(ii) Prove that the centre of an ellipse E is the mid-point of every chord which contains the centre.

(b) Let g be a similarity transformation having magnification ratio k. A triangle pqr is mapped onto a triangle $p'q'r'$ under g.

(i) Prove that $|\angle pqr| = |\angle p'q'r'|$ and using a similar argument show that the triangles pqr and $p'q'r'$ are equiangular.

(ii) Prove that if h is the incentre of the triangle pqr, $g(h)$ is the incentre of the triangle $p'q'r'$.