AN ROINN OIDEACHAIS M. 29

LEAVING CERTIFICATE EXAMINATION, 1996 22902

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 6 JUNE – MORNING, 9·30 to 12·00

Attempt **SIX QUESTIONS** (50 marks each)

Marks may be lost if all your work is not clearly shown or you do not indicate where a calculator has been used.

1. *(a)* Express $\frac{1-\sqrt{2}}{1+\sqrt{2}}$ in the form $a\sqrt{2}-b$, where $a, b, \varepsilon N$.

(b) *(i)* $(x + 1)$ is a factor of $x^3 + 5x^2 + kx - 12$.

Find the value of k and the other two factors of the cubic expression.

(ii) If $x = \sqrt{p} + \frac{1}{\sqrt{p}} + 1$ where $p > 0$, express $x^2 - 2x$ in terms of p.

(c) *(i)* Make a sketch of the region of the plane represented by

$$y \geq |x| \text{ and } y \leq 2 + |x|.$$

(ii) $x^2 - px + 1$ is a factor of $ax^3 + bx + c$ where $a \neq 0$.

Show $c^2 = a(a - b)$.

2. *(a)* Solve for x, y, z

$$x + y - z = 0$$
$$x - y + z = 4$$
$$x - y - z = -8.$$

(b) *(i)* Solve for x

$$\frac{2x - 7}{x + 3} < 1, \quad x \neq -3.$$

(ii) If $u_n = n!(n + 2)$ show that

$$(n + 1)u_n + (n + 1)! = u_{n+1}.$$

(c) Find the quadratic equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ given that $\alpha + \beta = 5$ and $\alpha\beta = k$, where $k \neq 0$.

Find the range of values of k for which the equation will have real roots.

3. (a) If $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$, find a matrix M such that $M = BA^{-1}$.

(b) $P(z) = (z - 2)(z^2 - 10z + 28)$.

(i) Plot on an Argand diagram the solution set of $P(z) = 0$.

(ii) Verify that the three points form an equilateral triangle.

(c) (i) $z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

where $i^2 = -1$.

Calculate $z_1 z_2$ in the form $x + iy$ where $x, y \in \mathbf{R}$.

(ii) $(2 + 3i)(a + ib) = -1 + 5i$. Express $a + ib$ in the form $r(\cos\theta + i\sin\theta)$ and hence, or otherwise, calculate $(a + ib)^{11}$.

4. (a) Find S_n the sum of n terms, of the geometric series

$$2 + \frac{2}{3} + \frac{2}{3^2} + \dots\dots\dots + \frac{2}{3^{n-1}}.$$

If $S_n = \frac{242}{81}$, find the value of n.

(b) (i) Show that $\frac{1}{\sqrt{n+1} + \sqrt{n}}$ is equal to $\sqrt{n+1} - \sqrt{n}$.

(ii) If $u_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$ find an expression for the sum of the first n terms in terms of n.

(c) $u_1, u_2, u_3, \dots\dots\dots u_n$ is a sequence, where $u_n = 1 + 2 + 3 + \dots\dots + n$.

(i) Show $u_n = \frac{n}{2}(n + 1)$.

(ii) Express $u_n - u_{n-1}$ in terms of n.

(iii) Show $u_n + u_{n-1} = n^2$.

(iv) Find $u_n^2 - u_{n-1}^2$.

Hence, show that

$$\sum_{1}^{n}\left(u_n^2 - u_{n-1}^2\right) = 1 + 2^3 + 3^3 + \dots + n^3 \text{ where } u_0 = 0.$$

5. *(a)* Solve the simultaneous equations

$$\log (x + y) = 2 \log x$$
$$\log y = \log 2 + \log (x - 1) \text{ where } x > 1,\ y > 0.$$

(b) *(i)* Write the binomial expansion of $(a + b)^4$ in ascending powers of b.

Find $\left(x + \frac{1}{x}\right)^4 - \left(x - \frac{1}{x}\right)^4$ in its simplest form.

(ii) Write u_{r+1}, the general term of the binomial expansion of $(3 + 2x)^n$ in terms of x, r and n.
If the coefficients of x^5 and x^6 are equal, find the value of n.

(c) Prove by induction that if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, $n \in N_0$.

6. *(a)* Differentiate

$$(i)\ \frac{2x}{x+1} \qquad (ii)\ 4e^{2x+1}$$

(b) *(i)* Find $\frac{dy}{dx}$ if $y = \ln \sqrt{x^2 + 1}$.

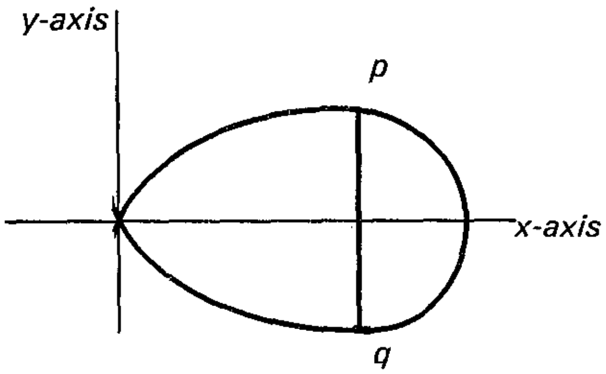
(ii) Take $x_1 = 1$ as the first approximation of a real root of the equation $x^3 - 2 = 0$. Find, using the Newton-Raphson method, x_2 and x_3 the second and third approximations. Write your answers as fractions.

(c) *(i)* $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ where a is a constant.

Show

$$1 + \left(\frac{dy}{dx}\right)^2 = \sec^2 \frac{\theta}{2}$$

(ii) $[pq]$ is a chord of the loop of the curve $y^2 = x^2(6 - x)$ so that the chord is parallel to the y-axis. Calculate the maximum value of $|pq|$.



7. *(a)* Find from first principles the derivative of x^2 with respect to x.

(b) The function f is defined

$$f : x \to (x-4)\{(x-3)^2 + 4\}.$$

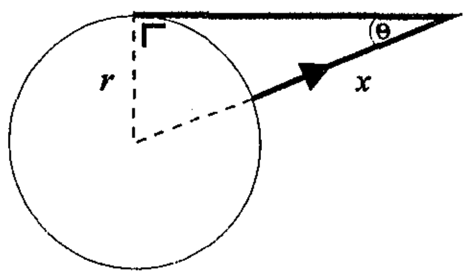
Find

(i) $f(3)$
(ii) the derivative with respect to x of the function at $x = 3$
(iii) the equation of the tangent at $(3, f(3))$.

Show that the tangent and the graph of $x \to f(x)$ will both intersect the x-axis at the same point.

(c) *(i)* Given $\tan y = x$, show $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$ and hence, find $\frac{d}{dx}\tan^{-1} x$.

(ii) An astronaut is at a height x km above the earth, as shown. He moves vertically away from the earth's surface at a velocity $\frac{dx}{dt}$ of $\frac{r}{5}$ km/h where r is the length of the earth's radius. He observes the angle θ as shown.



Express x in terms of r and θ.

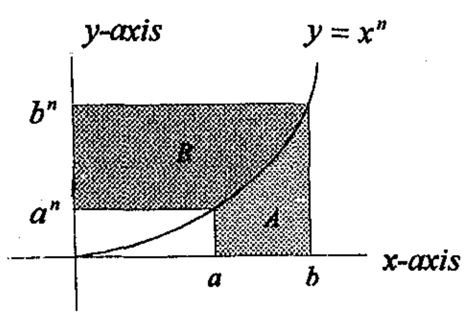
Hence find $\frac{d\theta}{dt}$ when $x = r$.

(a) Find *(i)* $\int \frac{1}{x^2}\, dx$ *(ii)* $\int (2x-1)^2\, dx$.

(b) Evaluate *(i)* $\int_0^2 \frac{dt}{\sqrt{4-t^2}}$ *(ii)* $\int_0^{\frac{\pi}{3}} \sin 2\theta \cos\theta\, d\theta$.

(c) *(i)* Calculate $\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\, dx$ to three places of decimals.

(ii) A is the area between the curve $y = x^n$, the x-axis and the lines $x = a$, $x = b$.



Calculate the area A in terms of a and b.

B is the area between the same part of the curve and the y-axis.
Determine the ratio

Area B : Area A.