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LEAVING CERTIFICATE EXAMINATION, 1995

MATHEMATICS—HIGHER LEVEL

PAPER 2 (300 marks)

FRIDAY, 9 JUNE — MORNING, 9·30 to 12·00

Attempt five questions from Section A and one question from Section B. Each question carries 50 marks.

Marks may be lost if necessary work is not shown or if you do not indicate where a calculator has been used.

SECTION A

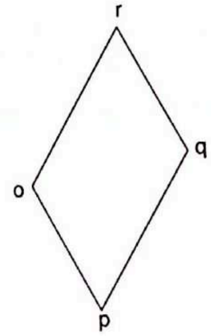
1. (*a*) Investigate if the point (100, 101) is on the line joining (2, 1) and (3, 2).

(*b*) $p\ (0, -2)$ and $q\ (-2, -6)$ are points of a circle centre $(k, -2k)$. Find k and write the equation of the circle.

(*c*) $x^2 + y^2 + 2x - 4y - 20 = 0$ is the equation of a circle. Two lines L and M intersect at $(2, -2)$. The distance from the centre of the circle to each line is $2\sqrt{5}$.
Find the equation of L and the equation of M.

2. (*a*) If $\vec{c} = 3\vec{i} - 2\vec{j}$ and $\vec{d} = 9\vec{i} + 6\vec{j}$, find a unit vector perpendicular to $\overrightarrow{cd}$.

(*b*) (i) $opqr$ is a parallelogram where o is the origin, $\vec{p} = 2\vec{i} - 8\vec{j}$ and $\vec{q} = 11\vec{i} + \vec{j}$. Express $\vec{r}$ in terms of $\vec{i}$ and $\vec{j}$.



(ii) $s \in [pq]$ and $|ps| : |sq| = 5 : 4$.
Express $\vec{s}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) If $\vec{w} = x\vec{i} - 4\vec{j}$ and $|\overrightarrow{ws}| = \sqrt{37}$, find two values for x.

Show that the larger value gives $\overrightarrow{ws} \perp \overrightarrow{rs}$.

Show that the smaller value gives an angle of $\cos^{-1}\left(\frac{-12}{37}\right)$ between $\overrightarrow{ws}$ and $\overrightarrow{rs}$.

OVER →

3. (*a*) Find the equation of the line through the point of intersection of

$$x - 2y + 6 = 0$$
and $$3x + 10y - 2 = 0$$

and which contains the point $(\frac{1}{4}, 0)$.

(*b*) f is the transformation $(x, y) \to (x', y')$ where

$$x' = 3x - y$$
$$y' = x + 2y.$$

For points $p\ (0, 0)$, $q\ (1, 0)$ and $r\ (0, 2)$ find $f(p)$, $f(q)$ and $f(r)$. Investigate if

(i) $|qr| = |f(q)f(r)|$.

(ii) the area of the triangle pqr is equal to the area of the triangle $f(p)f(q)f(r)$.

(*c*) L is a line with equation $ax + by + c = 0$. Prove that the image $f(L)$ is also a line, where f is the transformation which is given in (*b*).

M is a line with equation $ax + by + d = 0$. Deduce the equation of $f(M)$. Show $f(L) \parallel f(M)$.

4. (*a*) Evaluate $$\lim_{x \to 0} \left\{ \frac{\sin 5x}{x} + \frac{\sin 3x}{x} \right\}.$$

(*b*) When $\sin \alpha = \frac{5}{13}$ and $\cos \beta = \frac{4}{5}$, α and β less than $90°$, express $\sin(\alpha + \beta)$ in the form $\frac{a}{b}$ where $a, b \in Q$.

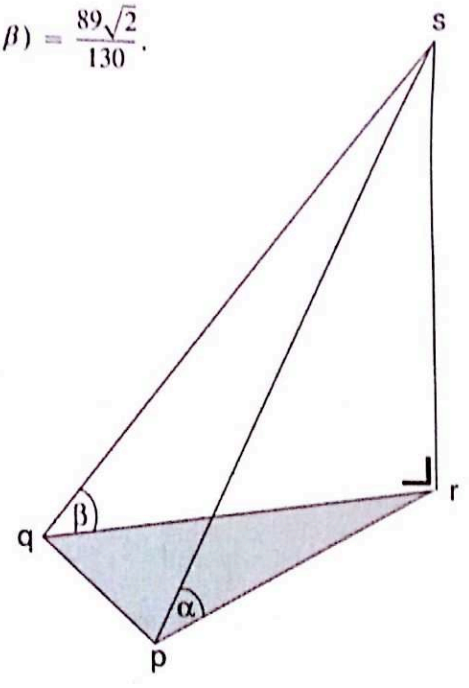
Hence, or otherwise, show

$$\cos(45 - \alpha - \beta) = \frac{89\sqrt{2}}{130}.$$

(*c*) p, q and r are points on level ground. $[sr]$ is a vertical tower of height h. The angles of elevation of the top of the tower from p and q are α and β, respectively.

(i) If $|\alpha| = 60°$ and $|\beta| = 30°$, express $|pr|$ and $|qr|$ in terms of h.

(ii) Find $|qp|$ in terms of h, if $\tan \angle qrp = \sqrt{8}$.

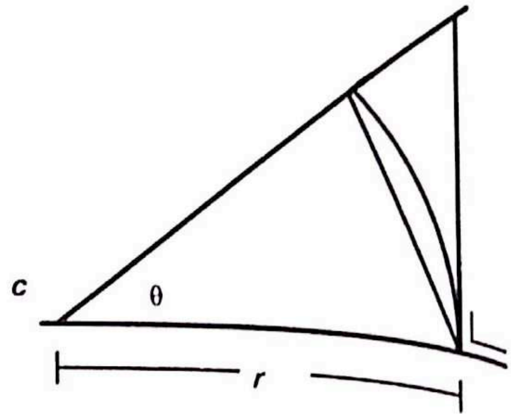


5. (*a*) Show that $\cos 2\theta = 2\cos^2\theta - 1$.

(*b*) Find the values of θ, $0 \leqslant \theta \leqslant 2\pi$, if $4\cos^2\theta - 3\sin\theta - 3 = 0$.

(*c*) (i) Prove that $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$.

(ii) A triangle is inscribed in a sector of a circle, centre c, radius r, $\theta < 90°$. A right-angled triangle circumscribes the sector (see diagram).



If the area of a sector is $\frac{1}{2}r^2\theta$

[Tables p. 7] prove

$$\sin\theta < \theta < \tan\theta.$$

6. (*a*) Five points are marked on a plane. No three of them are collinear. How many different triangles can be formed using these points as vertices?

Two of the five points are labelled x and y respectively. How many of the above triangles have [xy] as a side?

(*b*) (i) A bag contained 8 red, 12 blue and an unknown number of green beads. In a random draw the probability of drawing a green bead was $\frac{1}{5}$. How many green beads were in the bag at the start?

(ii) In how many ways can the letters of the word METCALF be arranged if M is always at the beginning and A and E are always side by side?

(*c*) Find $\sigma(x)$, the standard deviation of $0, x, 1$.

Show that $\sigma(x) = \sigma(1 - x)$ for all $x \in \mathbf{R}$.

Show that in $0 \leqslant x \leqslant 1$, the minimum value of $\sigma(x)$ is $\frac{1}{\sqrt{6}}$.

OVER →

7. (*a*) The weighted mean of the data below is IR£0.90.

Item	A	B	C	D	E
Price in IR£	1.40	0.80	0.30	3.50	0.50
Weight	3	5	x	1	3

Find the value of x.

(*b*) Solve the difference equation

$u_{n+1} - 2u_n - 53u_{n-1} = 0, n \geqslant 1$

when $u_0 = 0$ and $u_1 = 6$.

(*c*) Nine discs were each given a natural number from two to ten inclusive, each number different from the others. All nine were placed in a box.

(i) A disc was picked at random and replaced. A disc was then picked. Find the probability that both discs showed prime numbers.

(ii) Three discs were picked at random. What was the probability that three odd numbered discs or three even numbered discs were picked?

SECTION B

Do ONE question

8. (*a*) $f(x) = f(o) + \frac{f'(o)\,x}{1!} + \frac{f''(o)\,x^2}{2!} + \frac{f'''(o)\,x^3}{3!} + \ldots$ is the Maclaurin series.

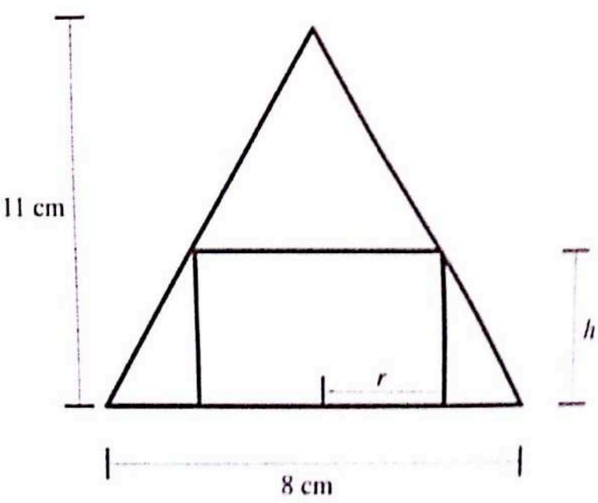
Write the Maclaurin series for $f(x) = e^x$ up to the term containing x^4. Hence find an approximation for e correct to four decimal places.

(*b*) Evaluate $\int_o^e x \ln x \, dx$.

(*c*) A right circular cone 11 cm in height and of base diameter 8 cm is to enclose a cylinder (see two dimensional diagram).

Express the height (h) of the cylinder in terms of its radius (r).

Find the maximum volume of the cylinder in terms of π.



9. (*a*) There were 9 white marbles in a bag. In a second bag there were 7 white and 2 black marbles. If a bag and then a marble were selected at random what was the probability of a black marble?

(*b*) There were 100 discs in a bag each having one of the one hundred natural numbers from 1 to 100 printed on it. There was a different number on each disc. Forty numbers were red. The rest were black. Twenty-six of the black numbers were even.

(i) How many discs had even red numbers?

(ii) If a disc were selected at random what was the probability that its number was odd given that it was red?

(iii) A disc was drawn and replaced. Then a disc was drawn. Find the probability that the first had a red number given that it was odd and that the second had an odd number given that it was black.

(*c*) A novelty shop advertised a biased (trick) die which would not give the expected number of fours as would a fair die. If the die gave 59 fours in 500 rolls, investigate at the 5% level of significance, whether or not the die was indeed biased.

10. (*a*) (i) Complete the table shown so that it will be the Cayley Table of a group. Give a reason for your answer.

*	0	1	
0	0	1	
1	1		

(*b*) In a group $G, *$ with identity e, prove

(i) $e^{-1} = e$.

(ii) $(a^{-1})^{-1} = a$ in all cases.

(iii) $(ab)^{-1} = b^{-1}a^{-1}$ in all cases.

(*c*) The group $S, \circ$ may be represented by the following table:

$\circ$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
σ_1	σ_1	σ_2	σ_3	σ_0	σ_5	σ_6	σ_7	σ_4
σ_2	σ_2	σ_3	σ_0	σ_1	σ_6	σ_7	σ_4	σ_5
σ_3	σ_3	σ_0	σ_1	σ_2	σ_7	σ_4	σ_5	σ_6
σ_4	σ_4	σ_7	σ_6	σ_5	σ_0	σ_3	σ_2	σ_1
σ_5	σ_5	σ_4	σ_7	σ_6	σ_1	σ_0	σ_3	σ_2
σ_6	σ_6	σ_5	σ_4	σ_7	σ_2	σ_1	σ_0	σ_3
σ_7	σ_7	σ_6	σ_5	σ_4	σ_3	σ_2	σ_1	σ_0

(i) Show that $S, \circ$ is non-commutative.

(ii) Find the order of σ_1 and list the elements in subgroup $\langle \sigma_1 \rangle, \circ$

(iii) Let C be the set of all elements which commute with σ_4.

Show that $\langle \sigma_1 \rangle, \circ$ is not isomorphic to $C, \circ$

(iv) Solve for k : $\sigma_4 \sigma_k \sigma_5 = \sigma_6$.

11. (*a*) Find the equation of the ellipse with foci $\left(\frac{-4}{3}, 0\right)$ and $\left(\frac{4}{3}, 0\right)$ and with eccentricity $\frac{4}{5}$.

(*b*) (i) L and M are perpendicular lines and f is a similarity transformation. Prove $f(L) \perp f(M)$.

(ii) Prove that the circumcentre of a triangle is mapped onto the circumcentre of the image triangle under a similarity transformation.

(*c*) (i) Show that the ratio of the areas of two triangles is an invariant under every affine transformation.

(ii) Show that for an ellipse E, the parallelogram formed by tangents at the end points of a pair of conjugate diameters has constant area.