AN ROINN OIDEACHAIS **M. 29**

LEAVING CERTIFICATE EXAMINATION, 1995 24529

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 8 JUNE – MORNING, 9.30 to 12.00

Attempt **SIX QUESTIONS** (50 marks each)

Marks may be lost if all your work is not clearly shown or if you do not indicate where a calculator has been used.

1.

(a) If $2x^2 + 5x + 6 = p(x+q)^2 + r$, for all x, find the value of p, of q and of r.

(b) If $P(x) = 6 + x - 4x^2 + x^3$, show that $(3 - x)$ is a factor of $P(x)$. Find the other two factors of $P(x)$.

(c) *(i)* Solve the inequality

$$\frac{5-x}{x-2} < 1, \quad x \neq 2 \text{ and } x \in \mathbf{R}.$$

(ii) Sketch the region which satisfies

$$y \geq 0 \text{ and } y \leq 3 - |x|.$$

2.

(a) Solve the simultaneous equations

$$y = 2x - 1$$

$$3x^2 - 2xy + y^2 = 9.$$

(b) *(i)* If α and β are the roots of the equation $x^2 - px + q = 0$, show that

$$(\alpha - \beta)^2 = p^2 - 4q.$$

(ii) Let $f(x) = \left(\frac{b^n - a^n}{b - a}\right)x + ab\left(\frac{a^{n-1} - b^{n-1}}{b - a}\right)$, *for* $a \neq b$.

Show that $f(a) = a^n$.

(c) For a fixed $k > 0$, let

$$f(x) = x^3 - k^2x, \quad x \in \mathbf{R}.$$

For $p \neq q$, divide $f(q) - f(p)$ by $q - p$.

Prove that if $0 \leq p < q \leq \frac{k}{\sqrt{3}}$, then $f(q) < f(p)$,

whereas,

if $\frac{k}{\sqrt{3}} \leq p < q$, then $f(q) > f(p)$.

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3. *(a)* The complex number $u = 4 + 3i$, where $i^2 = -1$.
Find the complex number $v = p + qi$, $p, q \in \mathbf{R}$, where

$$uv = 10 - 5i.$$

(b) Using De Moivre's theorem, prove that

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta .$$

(c) Let $M = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 7 & 10 \\ 21 & 23 \end{pmatrix}$.

(i) Simplify $M^{-1}A$.

(ii) If $MB = 2M + A$, express B in matrix form.

(iii) Express the simultaneous equations

$$\begin{aligned} x - 2y &= 3 \\ 3x + y &= -1 \end{aligned}$$

in matrix form and use matrix methods to solve them.

4. *(a)* Solve $\binom{n+2}{2} = 36$, for $n \in \mathbf{N}$.

(b) Show that $\dfrac{2}{n(n+2)} = \dfrac{1}{n} - \dfrac{1}{n+2}$.

If $u_n = \dfrac{2}{n(n+2)}$, find $\displaystyle\sum_{n=1}^{\infty} u_n$.

(c) *(i)* Write the first three terms in ascending powers of b and the general term in the binomial expansion of $(a + b)^n$ for $n \in \mathbf{Z}^+$.

If h is a constant and hx^3y^4 is a term in the expansion of $(2x + 5y^2)^n$, find the value of n and the value of h.

(ii) If $S_n = \dfrac{1}{n}\sqrt{1 + 2 + 3 + \bullet\bullet\bullet + n}$, find $\lim_{n \to \infty} S_n$.

5. (a) The first three terms of an arithmetic sequence are $6, -9$ and x. The first three terms of a geometric sequence are -9, x and y. Find the value of x and the value of y.

(b) (i) Find the value of x in

$$\log_2(x + 2) + \log_2(x - 2) = 5 .$$

(ii) If $2^x + 2^{1-x} - 3 = 0$, solve for x.

(c) Prove, using the method of induction, that if $r > 0$,

$$\frac{1}{(1 + r)^n} \leq \frac{1}{1 + nr}, \text{ for all } n \geq 1 .$$

Deduce that $\lim_{n \to \infty} x^n = 0$, if $0 < x < 1$.

6. (a) Find the derivative of the functions

(i) $(4x - 1)^3$

(ii) $\dfrac{x}{x^2 + x + 1}$.

(b) Find the derivative of the functions

(i) $x^2 \log_e (2x + 1)$, for $x > -\frac{1}{2}$

(ii) $\tan^{-1}\left(\frac{1}{x}\right)$, for $x \neq 0$.

(c) (i) The concentration C of an antibiotic in the bloodstream after a time of t hours is given by

$$C = \frac{5t}{1 + \left(\frac{t}{k}\right)^2} \text{ units },$$

where $k > 0$.

If the maximum concentration is reached at $t = 6$ hours, find the value of k.

(ii) Prove, from first principles, the product rule

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $u = u(x)$ and $v = v(x)$.

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7. *(a)* Find the slope of the tangent to the curve

$$4x^2 + 9y^2 = 40$$

at the point (1, 2).

(b) x_n is the n-th approximation to the positive root of $x^2 - 2 = 0$ and x_{n+1} is the next approximation.

Using the Newton-Raphson method, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

show that

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

If $x_0 = 1$, find x_2 correct to three places of decimals.

(c) Let $x = \frac{1}{2}(e^y - e^{-y})$.

Show that

$$y = \log_e\left(x + \sqrt{x^2 + 1}\right).$$

Hence, show that $\dfrac{dy}{dx}$ can be expressed in the form $\dfrac{p}{(1 + x^2)^q}$, $p, q \in \mathbf{R}$.

8. *(a)* Find (i) $\displaystyle\int (1 + 3x^2)dx$, (ii) $\displaystyle\int \cos 2x dx$.

(b) Determine the area enclosed by the curve $y = x^2 + 1$ and the line $y = 5$.

(c) *(i)* Evaluate

$$\int_0^{\pi/6} \cos^2 3\theta d\theta$$

and

$$\int_1^4 \frac{dx}{\sqrt{x}\,(1 + \sqrt{x})}.$$

(ii) Derive, by integration methods, the volume of a cone of vertical height h and base radius length r.