AN ROINN OIDEACHAIS — M. 30

LEAVING CERTIFICATE EXAMINATION, 1993 — 15339

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

FRIDAY, 11 JUNE - MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if necessary work is not clearly shown or if you do not indicate where a calculator has been used.

1.

(i) Show that $\sqrt{2x + 2} = x - 2$ when $x = 3 + \sqrt{7}$.

(ii) Solve $|2x - 1| = |x - 1|$, $x \in \mathbf{R}$.

(iii) The equations of two lines are $13x - 11y + 1 = 0$ and $17x + 19y + 1 = 0$. Find the equation of the line through the origin and their point of intersection.

(iv) The points a and b have coordinates $(1, 2)$ and $(5, -6)$ respectively. Find the equation of the locus of a point p where $|\angle apb| = 90°$.

(v) (a, b, c) represents the permutation $a \to b,\ b \to c,\ c \to a,\ d \to d$ of the set $\{a, b, c, d\}$. f is the permutation $(1, 2, 3)$ of the set $(1, 2, 3, 4)$. Write $f \circ f$ in the form (a, b, c).

(vi) Find the matrix of the symmetry in the line $x + y = 0$.

(vii) If $\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}^4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find a, b, c, d.

(viii) Find the period of the function $f : x \to \sin \frac{3}{2}x + \cos \frac{2}{3}x,\ x \in \mathbf{R}$.

(ix) Show that $(1, 0)$ is the centre of symmetry of the curve $x = t$, $y = \pm \sqrt{2t - t^2}$.

(x) I is the identity of a group G under multiplication. $g \in G$ is such that $g^3 = g^{10} = I$. Show that $g = I$.

OR

(x) Write the equation of the parabola which has vertex $(-1, 1)$ and focus $(-2, 1)$.

OVER→

2. *(a)* $3x^3 - 6x^2 + 5x - 1 = 0$ has roots α, β and γ.

(i) Write down the value of $\alpha\beta + \alpha\gamma + \beta\gamma$ and find the value of

$$\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2.$$

(ii) Hence find the equation which has roots $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$.

(b) Find the value of $a \in \mathbf{R}$ for which there is more than one solution in the simultaneous equations

$$\begin{aligned} ax + 3y - z &= 2 \\ 2x - 6y + 3z &= -1 \\ 3x - 3y + 2z &= a. \end{aligned}$$

Write two solutions for that value of a.

3. *(a)* Write the first three terms and the general term of the expansion of $(1 + x)^n$.

(b) Write, in ascending powers of x, the first three terms of

$\dfrac{1}{2 + x}$ and express each term in the form $\pm \dfrac{x^a}{2^b}$.

Hence, using a suitable value of x, or otherwise, evaluate $\displaystyle\sum_{r=0}^{\infty} \frac{1}{2^{r+1} \cdot 3^r}$.

(c) Prove by induction that $n^5 - n$ is divisible by 5, $n \in \mathbf{N}$.

4. *(i)* θ is the acute angle between the lines P: $x - y + 2 = 0$ and Q: $x - 2y - 1 = 0$.
Find $\tan\theta$.

θ is also the angle between a line L and the x-axis. L has positive slope and cuts the x axis at $(a, 0)$. Write the equation of L.

(ii) L cuts the lines P and Q. Write the coordinates of the points of intersection in terms of a.

(iii) Find the two values of a such that the area of the triangle formed by P, Q and L is 9, where the anticlockwise order of points is $P \cap Q$, $Q \cap L$ and $P \cap L$.

5. Show that if (x_1, y_1) is a point on each of the circles:

$S_1 = x^2 + y^2 + 4x - 3y + 2 = 0$ and $S_2 = 2x^2 + 2y^2 + 6x + 2y - 3 = 0$

then it is also a point on the circle $S_2 + \lambda S_1 = 0$.

Find the centre and the radius of *(i)* $S_1 = 0$ and *(ii)* $S_2 = 0$.

Hence, or otherwise, show that the circles intersect.

Two circles touch the y - axis and contain the points of intersection of $S_1 = 0$ and $S_2 = 0$. Find the equations of the circles.

6. Let $\mathbf{M} = \begin{pmatrix} 2 & 1\frac{1}{2} \\ 1\frac{1}{2} & -2 \end{pmatrix}$. Show that the equation $(x \;\; y)\; M \begin{pmatrix} x \\ y \end{pmatrix} = 0$

represents a pair of lines. Sketch the pair of lines.

B is the matrix of the rotation which maps the x-axis onto the line $x - 3y = 0$.

Find the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = B^{-1} MB.$$

Noting that $\begin{pmatrix} 3t \\ t \end{pmatrix}$ represents a point on $x - 3y = 0$, $t \in \mathbf{R}$,

show that $M \begin{pmatrix} 3t \\ t \end{pmatrix}$ also represents a point on $x - 3y = 0$.

Show also that $(3t, t)$ is equidistant from the lines represented by

$$(x\; y)\; M \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

OVER→

7. (a) Prove that

(i)
$$\sin x = \frac{2\tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}$$

(ii)
$$\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \tan\tfrac{1}{2} x$$

(b) (i) Where $\sin x \neq 0$, find the values of y in the range $0 < y < 2\pi$ such that $\sin(x + y) + \sin(x - y) = \sin x$.

(ii) Find the values of x in $0 < x < \frac{2\pi}{3}$ for which $\cos x = \cos 2x + \cos 4x$.

8. (a) Show that $G = \{1, 3, 5, 7\}$ is a group under multiplication mod 8.

Show that $H = \{1, 3, 7, 9\}$ is a group under multiplication mod 10.

Find the elements x of G and the elements y of H, if any, such that

$G = \{x^n \mid n \in N\}$ and $H = \{y^n \mid n \in N\}$.

Prove that G is not isomorphic to H.

(b) $J = \{1, 2\}$ under multiplication mod 3. Construct a Cayley table for J.

$K = \{(a, b) \mid a, b \in J\}$. An operation $*$ is defined in K such that for every (x, y) and (m, n) elements of K, $(x, y) * (m, n) = (xm, yn)$ where xm denotes the product of x and m in J and similarly for yn.

Given that the operation $*$ is associative investigate whether K is a group under the operation $*$.

OR

8. (i)

If $(at_1^2, 2at_1)$, $t \neq 0$ is one extremity of a focal chord of a parabola $y^2 = 4ax$, verify that the coordinates of the other extremity are

$$\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$$

(ii) Show that the tangents at the extremities of a focal cord of a parabola intersect on the directrix.

(iii) $y = 2$ is the axis of a parabola. $(5, 4)$ is a point on the parabola.
$x - 2y + 3 = 0$ and $8x + 4y - 31 = 0$ are tangents at the extremities of a focal chord.

Find the coordinates of the focus and hence the equation of the parabola.