AN ROINN OIDEACHAIS — **M. 29**

LEAVING CERTIFICATE EXAMINATION, 1993

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 10 JUNE – MORNING, 9·30 to 12·00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown or if you do not indicate where a calculator has been used.

1.

(i) Differentiate the function $\log_e\left(\dfrac{1+x}{1-x}\right)$ and express your answer in the form

$$\frac{p}{q - x^n}, \quad p, q, n \in \mathbf{R}.$$

$\ln(1+x) - \ln(1-x)$

$\frac{1}{x+1} + \frac{-1}{x-1} = \frac{x-1-x-1}{x^2+1}$

$= \frac{-2}{1-x^2}$

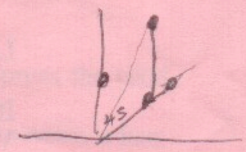
(ii) Evaluate at $x = \pi/3$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}.$$

$\cos\left(\frac{\pi}{3}\right)$

(iii) On an Argand diagram, mark the points

$$z_1 = i, \quad z_2 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \quad z_1 + z_2.$$



Hence, or otherwise, deduce that $\tan\dfrac{3\pi}{8} = \sqrt{2} + 1$

(iv) Evaluate

$$\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1 - 4x^2}}.$$

(v) Find the area of the bounded region enclosed by the curve $y = x^3$, the X axis and the line $x = 2$.

(vi) Let a_n be the n th term of a sequence and $a_{n+1} = \dfrac{2}{3}a_n + \dfrac{8}{3a_n^2}$,

where $a_1 > 0$.

If $\lim\limits_{n \to \infty} a_n = L$, write down $\lim\limits_{n \to \infty} a_{n+1}$ and hence find L.

(vii) Let $f(x) = \frac{1}{2}(e^x - e^{-x})$, where $f'(x) = \dfrac{d}{dx} f(x)$.

Show that $f(x).f'(x) = \frac{1}{2} f(2x)$.

(viii) Evaluate the limit

$$\lim_{x \to \infty} \frac{2x^2 + 3x + 4}{x^2 - 2x - 3}.$$

(ix) By using the Comparison test, or otherwise, establish the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 1}{(n^2 + n)\,2^n}.$$

(x) Two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = |\vec{b}| = |\vec{a} - \vec{b}|$.

Show that $\vec{a}.\vec{b} = \frac{1}{2}|\vec{a}|^2$.

OR

(x) When a biased dice is thrown, a score of 6 is twice as likely as a score of 5, a score of 5 is twice as likely as a score of 4 and scores of 1, 2, 3, 4 are equally likely. Calculate the probability of scoring an even number.

2. *(a)* Let $z = x + iy$, where $x, y \in \mathbf{R}$ and $i = \sqrt{-1}$.

A is the set of points represented by $|z| = 5$ and B is the set of points represented by $z + \bar{z} = 8$.

Plot A and B on the Argand diagram.

Determine z_1 and z_2, the points of intersection of A and B.

(b) Express $z = \dfrac{\sqrt{3} + i}{\sqrt{3} - i}$ in the form $\cos\theta + i\sin\theta$.

Find the least positive value of n such that z^n is a real number.

(c) Let $z = x + iy$, where $x, y \in \mathbf{R}$ and $i = \sqrt{-1}$.

Indicate clearly on an Argand diagram in the z-plane the set K of z defined by

$$K = \{ z \mid x^2 + y^2 = 4\}.$$

Let $w = 2z^2$.

If $w = u + iv$, express x and y in terms of u and v.
Verify that the image of K in the w-plane is the circle

$$u^2 + v^2 = 64.$$

3. *(a)* The curve

$$y = \frac{p + qx}{x(x + 2)}, \quad p, q \in \mathbf{R}, x \neq 0, x \neq -2$$

has zero slope at the point $(1, -2)$.

Show that $p = 2$ and find the value of q.

(b) $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ is a geometric series with common ratio r, where $|r| \neq 1$, $r \neq 0$ and $u_1 \neq 0$.

(i) Express the series

$$u_1 u_2 + u_2 u_3 + u_3 u_4 + \ldots + u_n u_{n+1}$$

in terms of u_1 and r.

Hence, show that its sum is $\dfrac{u_1^2 r(1 - r^{2n})}{1 - r^2}$.

(ii) Find the sum of the series

$$\frac{u_1}{u_1 + u_2} + \frac{2u_2}{u_2 + u_3} + \frac{3u_3}{u_3 + u_4} + \ldots + \frac{nu_n}{u_n + u_{n+1}}.$$

4. *f*

(a) Find the value of the derivative of the function

(i) $2 \sin x \cos x$ at $x = \pi/8$

(ii) $x^2 \log_e x$ at $x = e$

(b) If $y = \dfrac{2x - 1}{4} e^{2x}$,

calculate the value of $\dfrac{dy}{dx}$ when $x = \log_e 3$.

(c) Find the equation of the tangent to the curve

$$x^2 y - y^2 x = 6$$

at the point $(-2, 1)$.

(d) Let $x = a(\cos \phi + \phi \sin \phi)$ and $y = a(\sin \phi - \phi \cos \phi)$, where $a \neq 0$, $-\pi < \phi < \pi$ and $\phi \neq \pm \pi/2$.

Show that $\dfrac{dy}{dx} = \tan \phi$.

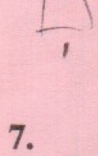
5. A cylindrical tin can, closed at both ends, has a given fixed volume, V. Denoting the radius and height of the tin can by r and h, respectively, express the total surface area, A, of the tin can in terms of V and r. Prove that the total surface area, A, of the cylindrical tin can will be a minimum when the height of the can equals the diameter of its base.

6. (a) Evaluate

(i) $$\int_0^1 (x^2 + 3)^2 \, dx$$

(ii) $$\int_3^8 \frac{x}{\sqrt{1 + x}} \, dx$$

(iii) $$\int_0^{\pi/4} (1 + \tan \theta) \sec^2 \theta \, d\theta$$



(b) The region bounded by the line $y = \frac{x}{3}$, the X axis and the line $x = 3$ is revolved about the X axis.
Find the volume of the solid generated.

$\frac{4}{3} \pi (1)^2 3$

7. (a) Let $$u_n = \frac{1}{1 + \left(\frac{2}{3}\right)^n}$$

Determine whether

(i) as $n \to \infty$, the sequence $u_1, u_2, \ldots, u_n, \ldots$ converges

(ii) the series $\displaystyle\sum_{n=1}^{\infty} u_n$ converges.

OVER →

$u_1 = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right]$

$u_2 = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right]$

$u_3 = \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right]$

$u_4 = \frac{1}{2}\left[\frac{1}{5} - \frac{1}{7}\right]$

7. *(b)* If $u_n = \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$, show that the series

$$\sum_{n=1}^{n} u_n = \frac{5}{12} - \frac{1}{2}\left[\frac{1}{n+2} + \frac{1}{n+3}\right].$$

u_n

Hence, or otherwise, deduce that the series $\sum_{n=1}^{\infty} u_n$ converges.

$\frac{1}{2}\left[\frac{1}{3} + \frac{1}{2}\right]$

(c) Find the values of $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ for which the series of positive terms

$\frac{4}{3}\sqrt{|\sin x|} = 1$

$$\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n \frac{\sin^{2n} x}{n+1} \text{ converges.}$$

8. *(a)* $\vec{x}$ and $\vec{y}$ are any two vectors with respect to o as origin.
Show that

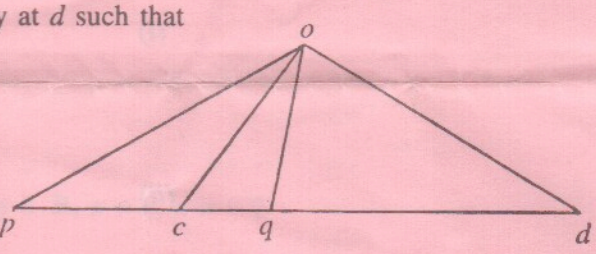
$$\vec{r} = t\left[\frac{\vec{x}}{|\vec{x}|} + \frac{\vec{y}}{|\vec{y}|}\right],$$

$t \in \mathbf{R}$, is a point on the bisector of $\angle xoy$.

(b) $[pq]$ is divided internally at c and externally at d such that

$$\vec{pc} = \frac{2}{3}\vec{pq} = \frac{1}{3}\vec{pd}$$

where o is the origin.



(i) Express $\vec{c}$ in terms of $\vec{p}$ and $\vec{q}$ and express $\vec{d}$ in terms of $\vec{p}$ and $\vec{q}$.

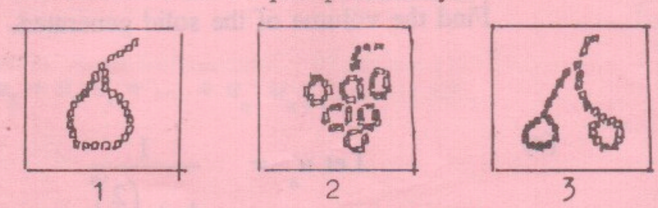
(ii) If $|\angle cod| = 90°$, use the scalar product to show that $|\vec{p}| = 2|\vec{q}|$

(iii) Deduce that oc bisects $\angle poq$.

OR

8. *(a)* A computer game has three windows, labelled 1, 2 and 3. Each window displays the same choice of five different fruit. All outcomes have equal probability.

Sample outcome in one game



(i) Calculate the probability of obtaining at least two identical fruit in one game.

(ii) To win a game at least two identical fruit are required.
Find the probability of winning just two out of three games.

(b) The mean and standard deviation of the diameters, measured in cm, of a random sample of 100 tomatoes are 6 and 1·166, respectively.
The distribution of tomatoes is assumed normal with this mean and standard deviation.
For packing purposes, 5% are rejected as too small and 5% are rejected as too large.
Find the range of size of tomatoes suitable for packing.