David Malone

AN ROINN OIDEACHAIS **M. 29**

LEAVING CERTIFICATE EXAMINATION, 1992

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 11 JUNE – MORNING, 9·30 to 12·00

14514

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown or if you do not indicate where a calculator has been used.

1.

10 *(i)* Let $f(x) = \left(\dfrac{1 + x + x^2}{1 - x + x^2}\right)$. Form a new function $g(x)$ by replacing x in $f(x)$ by $-x$. Show that $g(x) = 1/f(x)$.

10 *(ii)* Evaluate at $x = \pi/6$

$$\lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}$$ $-\frac{1}{2}$

10 *(iii)* Let $\dfrac{\sqrt{2}\,(\cos\theta - i\sin\theta)}{2 + i} = \dfrac{1 - 3i}{5}$, where $i = \sqrt{-1}$.

Show that $\cos\theta - i\sin\theta = \dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$.

Hence find θ where $0 \le \theta \le \dfrac{\pi}{2}$.

4 *(iv)* Using the substitution $u = \sin^{-1}\dfrac{x}{2}$, or otherwise, evaluate

$$\frac{1}{\pi^2}\int_0^1 \frac{\sin^{-1}\frac{x}{2}}{\sqrt{4 - x^2}}\,dx$$ $\frac{1}{36}$

10 *(v)* Find the area of the bounded region enclosed by the curve $y^2 = x$ and the line $x = 9$. 36.

10 *(vi)* Let a_n be the nth term of a sequence and $a_{n+1} = \frac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$, where $a_1 > 0$.

If $\lim\limits_{n\to\infty} a_n = L$, write down $\lim\limits_{n\to\infty} a_{n+1}$. Hence find L. $L = \sqrt{2}$

7/9 *(vii)* Let $f(x) = \frac{1}{2}(e^x - e^{-x})$ and $f'(x) = \dfrac{d}{dx}f(x)$.

Find values for e^x which satisfy $5f'(x) + f(x) = 7$. $e^x = \dfrac{7 \pm \sqrt{73}}{2}$

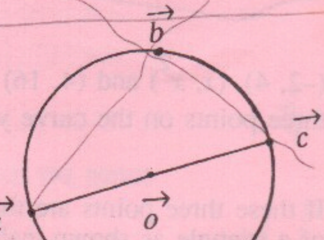
10 *(viii)* Show $\dfrac{1}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}$.

If $u_n = \dfrac{1}{\sqrt{n+1} + \sqrt{n}}$, evaluate $\displaystyle\sum_{n=1}^{15} u_n$. 3.

10 *(ix)* Show clearly that $f(x) = \sin x - x\cos x$ is an increasing function in the domain $-\pi < x < \pi$.

16 *(x)* A circle with centre $\vec{o}$ as origin has a diameter with end points $\vec{a}$ and $\vec{c}$. Using the fact that the angle in a semicircle is a right angle, show that for any point $\vec{b}$ on the circumference,

$\vec{c}\,.(\vec{b} - \vec{a}) + \vec{a}\,.\vec{b} = |\vec{b}|^2$.



OR

4/4 *(x)* A drawer contains 5 red markers and n blue markers. One marker is drawn at random and not replaced. Another marker is then drawn at random. If the probability that both markers are blue is 1/6, how many blue markers were in the drawer?

OVER→

$\dfrac{n}{n+5} \times \dfrac{n-1}{n+4} = \dfrac{1}{6}$ $\quad \dfrac{n^2-n}{n^2+9n+20} = \dfrac{1}{6}$ $\quad n^2+9n+20 = 6n^2-6n$ $\quad 0 = 5n^2-15n-20$ $\quad (n+1)(n-4)$ $\quad n=4$

2. (a) Determine the real numbers p and q so that $(p + iq)^2 = 5 + 12i$, where $i = \sqrt{-1}$. $p = 3$, $q = 2$

(b) If $z = \dfrac{1 + i\tan\theta}{1 - i\tan\theta}$, show that $|z| = 1$.

(c) Let $K = \{ z \mid x^2 + y^2 = 1\}$, where $z = x + iy$, $x, y \in \mathbf{R}$.

On an Argand diagram in the w-plane plot $f(K)$, the image of K, under the transformation

$$w = f(z) = 2i\,(z + 3).$$

$R = 2$
$c = (0, 6)$

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3. (a) Investigate if the function

$$f(x) = x^2 \sin x$$

has a local maximum, a local minimum or a point of inflection at $x = 0$.

No max/min
Yes Point of Inflection

10 min

(b) Let $u_n = ar^{n-1}$, where $r \neq 1$ and $S_n = u_1 + u_2 + u_3 + \dots + u_n$.

Find in terms of a, r and n

(i) S_n $\qquad \dfrac{a(1-r^n)}{1-r}$

(ii) $u_1 . u_2 . u_3 . \dots . u_n$ $\qquad a^n r^{\frac{(n-1)(n)}{2}}$

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4. (a) Find the value of the derivative of the function

(i) $\left(x + \dfrac{1}{x}\right)^2$ at $x = 2$. $\qquad \dfrac{15}{4}$

(ii) $x\,4^x$ at $x = 1$ $\qquad 4 \ln 4 + 4$

(b) Let $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, where $a \neq 0$ and $-\pi < \theta < \pi$.

Show that

$$\frac{dy}{dx} = \cot\frac{\theta}{2}.$$

17 min

(c) A tangent line is drawn to the curve $y = \sin^{-1} x$ at the point $\left(\dfrac{1}{2}, \dfrac{\pi}{6}\right)$. Find where this line meets the X axis.

$m = \dfrac{2}{\sqrt{3}}$

$\left(\dfrac{1}{2} - \dfrac{\pi}{4\sqrt{3}}, 0\right)$

(d) If $x^3 + y^3 = 3xy$, find $\dfrac{dy}{dx}$ at the point $\left(\dfrac{3}{2}, \dfrac{3}{2}\right)$. $\qquad -1$

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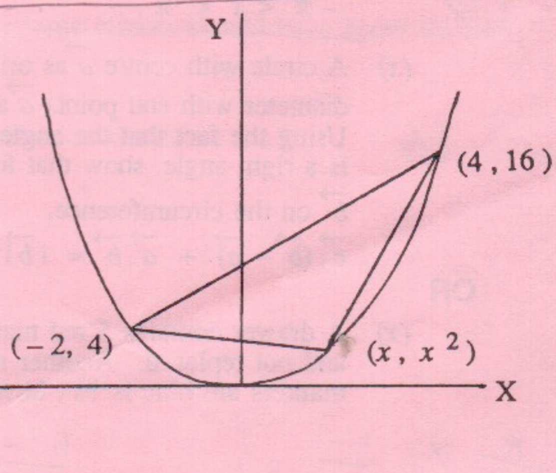
5. $(-2, 4)$, (x, x^2) and $(4, 16)$ are three points on the curve $y = x^2$.

If these three points are the vertices of a triangle as shown, calculate the value of x so that the area enclosed by the triangle is a maximum.

$x = 1$



6. Evaluate

$\frac{4}{3}$ (i) $$\int_1^4 (2 - \sqrt{x})\, dx$$

7 min. (+mistake)

$\frac{1}{3}$ (ii) $$\int_0^{\pi/2} \sin x \cos^2 x\, dx$$

Ln 2 (iii) $$\int_3^4 \frac{2x - 6}{x^2 - 6x + 10}\, dx$$

$\frac{1}{e+1} - \frac{1}{2}$ (iv) $$\int_0^1 \frac{e^{-x}}{(1 + e^{-x})^2}\, dx .$$

$u = 1 + e^{-x}$ $\quad \frac{du}{dx} = -e^{-x}$

$\int_2^{1+e^{-1}} \frac{e^{-x}}{u^2} \frac{du}{-e^{-x}}$

u^{-2}

$\int_{1+\frac{1}{e}}^{2} \frac{1}{u^2}\, du$

$\left[-\frac{1}{u}\right]_{1+\frac{1}{e}}^{2}$

$-\frac{1}{2} + \frac{1}{1+\frac{1}{e}}$

$\frac{-(1+\frac{1}{e}) + 2}{2(1+\frac{1}{e})}$

$= \frac{1 - \frac{1}{e}}{2(1+\frac{1}{e})}$

7. (a) Let $$u_n = \frac{(n + 1)(n + 2)}{(n + 3)(n + 4)}$$

Determine whether

(i) as $n \to \infty$, the sequence $u_1, u_2, \ldots, u_n, \ldots$ converges

Yep → 1

(ii) the series $$\sum_{n=1}^{\infty} u_n$$ converges.

No quickey

(b) Use the Comparison test to verify that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n} - 1}{n^2 + 4n}$$

converges.

(c) Find the values of $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for which the series

$-\frac{\pi}{3} < \alpha < \frac{\pi}{3}$

$$\sum_{n=1}^{\infty} \frac{\sec^n \alpha}{2^n (n + 1)}$$

converges.

$-\frac{\pi}{2}$ $\quad \frac{\pi}{2}$ $\quad .231058$ $\quad -0.2310$

$-.2310585579$

OVER→

8. *(a)* If r divides $[ab]$ internally in the ratio $m{:}n$, show that

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}.$$

$pqrs$ is a parallelogram. The mid-point of $[pq]$ is c.

Verify that $\vec{pr}$ and $\vec{sc}$ meet in a common point of trisection.

(b) r_1 is a point of the line ab and

$$\vec{r_1} = \vec{a} + t\,(\vec{b} - \vec{a}),\ t \in \mathbf{R}.$$

Find $\vec{r_1}$ if $\vec{a} = 4\vec{i} + 3\vec{j}$ and $\vec{b} = 8\vec{i} + \vec{j}.$

r_2 is a point of the line pq and

$$\vec{r_2} = 3\vec{i} + 5\vec{j} + k\,(\vec{i} + \vec{j}),\ k \in \mathbf{R}.$$

(i) Calculate the cosine of the angle between ab and pq.

(ii) Find $\vec{s}$, where s is the point $ab \cap pq$.

OR

8. Hens' eggs have masses which may be said to have a normal distribution about a mean mass of 60 g and a standard deviation of 15 g.

Eggs of mass less than 45 g are classified as small.

The remainder are classified into two further divisions called standard and large.

(i) If an egg is picked at random from a batch, find the probability that it is small.

(ii) It is desired that the standard and large classes should have about the same number of eggs in each.

Estimate the mass at which this division should be made.