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LEAVING CERTIFICATE EXAMINATION, 1991

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

FRIDAY, 7 JUNE - MORNING, 9.30 to 12.00

300

80

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown or if you have not indicated where a calculator has been used.

$\cos x^2 . (2x$

1. (i) Differentiate the function $x \to (\sin x^2)^2$ and express your answer in the form $u \sin ux$.

$2(\sin x^2) \cos x^2 . 2x$

(ii) Solve the equation $2e^{2y} - 3e^{y} - 2 = 0$.

(iii) Evaluate

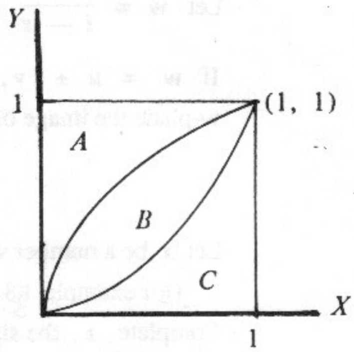
$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{d\sqrt{x}}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

(iv) Use the substitution $u = \tan^{-1}\frac{x}{3}$, or otherwise, to evaluate

$$\frac{1}{\pi^2}\int_0^3 \frac{\tan^{-1}\frac{x}{3}}{x^2+9}\,dx$$

(v) Section B in the diagram is bounded by the curves

$y = x^n$ and $y = x^{1/n}$.



Given that the three sections A, B, C are of equal area, find n.

(vi) If $\dfrac{1+i}{1-i} = k\left(\dfrac{1-i}{1+i}\right)$, $i = \sqrt{-1}$, find $k \in \mathbf{R}$.

(vii) On an Argand diagram mark the points

$$z_1 = 1, \quad z_2 = \frac{\sqrt{2}}{1-i}, \quad z_1 + z_2.$$

Hence, or otherwise, deduce that $\tan\frac{\pi}{8} = \sqrt{2} - 1$.

(viii) The sequence $u_1, u_2, \ldots, u_n, \ldots$ is such that

$4u_n = [1 + (-1)^n][1 + i^n]$ where $i = \sqrt{-1}$.

Write out the first four terms and calculate

$$\sum_{n=1}^{1000} u_n.$$

$(x^{n-1} + x^n)$

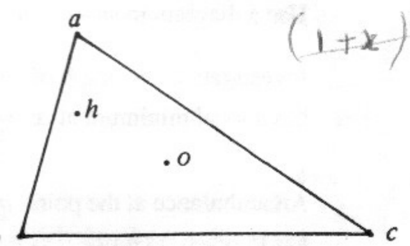
$$\frac{(1-x)+(x-x^2)+(x^2-x^3)+ \ldots + x^n}{1-x}$$

(ix) Express as a single fraction $1 + x + x^2 + \ldots + x^{n-1} + \dfrac{x^n}{1-x}$.

$\dfrac{1}{1-x}$

$(1+x)^{n-1}$

(x) In the triangle abc take the circumcentre, o, as origin.



If $\vec{h} = \vec{a} + \vec{b} + \vec{c}$ prove that

$$|\vec{ah}|^2 + |\vec{bc}|^2 = 4r^2$$

where r is the radius length of the circumcircle.

$|\vec{ah}|^2 + |\vec{bc}|^2$

$\vec{ah}\cdot\vec{ah} + \vec{bc}\cdot\vec{bc}$

$(b+c)\cdot(b+c) + (b-c)\cdot(b-c)$

$2b\cdot b + 2c\cdot c$

$2r^2 + 2r^2$

$4r^2$

OR (x) Three pupils have their birthdays in the same week. What is the probability that the birthdays fall on three different days?

2. (a)

(i) Express $\dfrac{1 + i\tan\frac{\pi}{6}}{1 - i\tan\frac{\pi}{6}}$ in the form $\cos\theta + i\sin\theta$.

(ii) Solve the equation $(1 + iz)(\sqrt{3} - i) = (1 - iz)(\sqrt{3} + i)$.

$2(\sin x^2)(\cos x^2)(2x)$ $\quad$ $2x(\sin 2x^2)$ $\quad$ $u = 2x$ $\quad$ $2x[\sin(2x.x)]$

(b) Let $z = x + iy$.

Indicate on an Argand diagram in the z-plane the half-plane K defined by

$$\{ z \mid x + y > 1 \}$$

Let $w = \dfrac{1}{i - z}$ where $z \neq i$.

If $w = u + iv$, express x and y in terms of u and v and hence indicate in the w-plane the image of K under the transformation

$$w = \frac{1}{i - z}, \quad (z \neq i).$$

3. Let x be a number which reads the same from left to right as it does from right to left. (for example 88, 313, 1991)

Complete x, the six digit number,

$a(10^5) + b(10^4) + c(10^3) + \ldots$

and show that if this number is divisible by 7, then

$6a + c$ is also divisible by 7.

Give an example of such a number and investigate if it is also divisible by 13.

$a\,10^5 + b\,10^4 + c\,10^3 + c\,10^2 + b\,10 + a$

$a(10^5+1) + b(10^4+10) + c(10^3+10^2)$

$+ 6a + 1c$

4. (a) (i) If $y = \log_e (x + \sqrt{x^2 + 1})$, find the $\dfrac{dy}{dx}$ at $x = \frac{3}{4}$ and express your answer in the form $\dfrac{a}{b}$ for $a, b \in \mathbf{N}_0$.

(ii) If

$$y = \frac{e^{2x} - 1}{e^{2x} + 1}$$

show that

$$\frac{dy}{dx} = 1 - y^2.$$

(b) The graph of $y = f(x)$ has a point of inflexion at $x = a$.

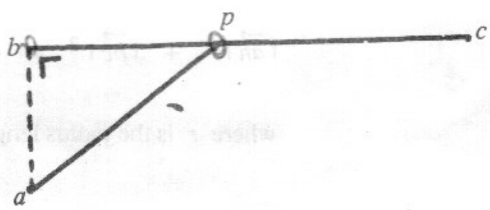
Use a diagram to show that $\dfrac{d^2y}{dx^2}$ changes sign in the neighbourhood of $x = a$.

Investigate if the graph of $y = x^3(10 - 3x^2)$ has a local minimum at $x = 0$ and verify that it has points of inflexion at $x = \pm 1$.

5. An ambulance at the point a has to reach, as quickly as possible, the point c on the road bc. It travels at 80 km/h between a and any point on the road. Once on the road it travels at 100 km/h.

If $|bc| = 9$ km, $|ba| = 3$ km and $|bp| = x$ km, derive in terms of x an expression for t, the time taken to reach c.

Find the value of x for which the time is minimum and calculate this minimum time.



6. (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos x \cos 3x \, dx$

(b) Show that for $x \in \mathbf{R}$

$$\frac{1}{(e^x + 1)^2} = 1 - \frac{e^x}{e^x + 1} - \frac{e^x}{(e^x + 1)^2}$$

and hence evaluate

$$\int \frac{dx}{(e^x + 1)^2}$$

(c) If $J_n = \int_0^1 x^n \sqrt{1 - x^2}\, dx$,

evaluate J_0 and J_3.

7. **(a)** Show that for $h > 0$ and $n \in \mathbf{N}$

$$(1 + h)^n > 1 + nh \quad \text{for } n \geq 2.$$

In addition if $\epsilon > 0$, show that

$$\frac{1}{(1 + h)^n} < \epsilon \quad \text{for } n > \frac{1 - \epsilon}{\epsilon h}.$$

Deduce that for $0 < a < 1$

$$\lim_{n \to \infty} a^n = 0.$$

$\frac{1}{1+nh} < \epsilon$

(b) Find the range of values of $x > 0$ for which

$$\sum_{n=1}^{\infty} \frac{2^{n+1} (x + 1)^n}{n^2 \cdot 3^n}$$

converges.

$1 < \epsilon + n\epsilon h$

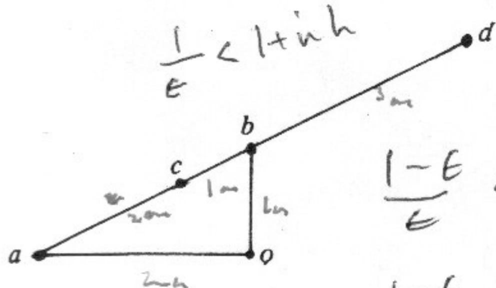
$\frac{1}{\epsilon} < 1 + nh$

8. The four points a, b, c, d are collinear.

Taking o as origin

$\vec{ac} = 2\vec{cb}$, $\vec{ad} = 2\vec{bd}$

and $|\vec{a}| = 2|\vec{b}|$



$\vec{c} = \frac{2\vec{b} + \vec{a}}{3}$

$\vec{d} = 2\vec{b} - \vec{a}$

$\frac{1-\epsilon}{\epsilon} < nh$

$\frac{1-\epsilon}{h\epsilon} < n$

(i) Express $\vec{c}$ and $\vec{d}$ in terms of $\vec{a}$ and $\vec{b}$.

(ii) Using scalar product, or otherwise, show

$$|\angle\, cod| = \frac{\pi}{2}.$$

$\vec{c} \cdot \vec{d}$

$= \frac{1}{3}(4b \cdot b - a \cdot a)$

$= \frac{1}{3}[4|b|^2 - |a|^2]$

$= \frac{1}{3}[|b|^2 - |2b|^2]$

$= 0$

$= \angle cod = \frac{\pi}{2}$

(iii) If r is a point on the bisector of $\angle\, aob$, show that

$$\vec{r} = k\,(|\vec{b}|\,\vec{a} + |\vec{a}|\,\vec{b}) \quad \text{where } k \in \mathbf{R}.$$

Is it possible to find a value of k for which $\vec{r} = \vec{c}$?

$a^n = (1 + (a-1))^n$

$> 1 + (a-1)n$

$n \to n$

$a^n > 1 + \emptyset$

$a^n > \emptyset$

OR

8. Z is a random variable having the standard normal distribution, $N(0, 1)$.

Use your Tables P.36 to find the value of z_1 for which

$$P(Z \geq z_1) = 0{\cdot}95.$$

A researcher captured a large number of pigeons in a certain region.

Each pigeon before being released was tagged with the message that when found a report of the finding be made to the researcher.

The probability of getting a report is 0·02 .

If the researcher wishes to be 95% certain of getting at least 20 reports, find the least number of pigeons that should be tagged.

(Note: If X is the random variable denoting the number of reports received, then $X \geq 20 - 0{\cdot}5$ should be used in the calculations).

$|a|\, b \cdot r = |b|\, a \cdot r$

$\frac{a \cdot r}{|r||a|} = \frac{b \cdot r}{|r||b|}$ $\quad |a||r|\cos\theta = |b||r|\cos\theta$