LEAVING CERTIFICATE EXAMINATION, 1991

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 6 JUNE - MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown
or if you have not indicated where a calculator has been used

1. (i) Find the positive root of $\left(1 - \frac{1}{x}\right)^2 = 2$ and write it in the form $a + \sqrt{b}$.

(ii) Find the value of $x > \frac{2}{5}$ for which $|x| = |2 - 5x|$.

(iii) Find the slopes of the lines which make an angle measuring 45° with the line

$$2x - 3y + 1 = 0.$$

(iv) Show that the line $3x - 4y - 26 = 0$ is a tangent to the curve

$$x^2 + y^2 - 2x + 4y - 4 = 0.$$

(v) If (a, b, c, d) denotes the permutation (map) $a \to b$, $b \to c$, $c \to d$, $d \to a$, express the permutation $(1, 3, 4, 2)$ in the form $\begin{pmatrix} 1 & 2 & 3 & 4 \\ p & q & r & s \end{pmatrix}$.

(vi) Find the equation of the image of the line $y = x + 3$ under the transformation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3$.

(vii) Show that for $k, a, x \in \mathbf{R}_0$

$$\left[k \begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix} \right]^{-1} = k^{-1} \begin{pmatrix} a & 0 \\ 0 & x \end{pmatrix}^{-1}.$$

(viii) Show that for $n \in \mathbf{Z}$

$$\frac{\sin[(2n - 1)\pi - x]}{\cos[2n\pi - x]} = \tan x.$$

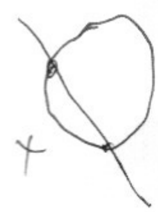
(ix) Let K be the set of points (x, y) defined by

$$x = 1 - \frac{1}{t}, \quad y = 1 + \frac{1}{t}, \quad t \in \mathbf{R} \text{ and } t \neq 0.$$

Let L be the line $x + y = 2$. Show that there is a point of L which is not in K.

(x) X, Y, Z, are the perpendiculars from the vertices of an equilateral triangle to the opposite sides. S_X, S_Y, S_Z, are the axial symmetries of the plane in the lines X, Y, Z, respectively. Show that $\{S_X, S_Y, S_Z\}$ is not closed under composition.

OR (x) Find the equation of the parabola which has its vertex at $(-2, 2)$, its focus on the X axis and has the line $x + 2 = 0$ as axis.



2. (a) If α, β are the roots of

$$3x^2 + 3x - 1 = 0,$$

find the quadratic equation which has α^2, β^2 as roots.

(b) The roots of

$$8x^3 - 12x^2 - 6x + 1 = 0$$

are in geometric sequence.
Find the three roots.

$16 - 20 + \frac{1}{2} = 0$

3. (a) Explain the notation $\binom{n}{r}$ and say why $\binom{n}{r}$ is also the number of subsets of size r which can be chosen from a set containing n elements.

$\boxed{n \mid n-1 \mid \cdots \mid n-r}$

Find $\sum\limits_{r=0}^{n} \binom{n}{r}$, the total number of subsets which can be chosen from a set containing n elements.

$\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n$

$\frac{n(n-1)(\cdots)(n-r+1)}{r!}$

(b) Show that $\binom{-3}{r} = (-)^r \dfrac{(r+1)(r+2)}{2}$.

Write in ascending powers of x the first four terms in the expansion of

$$\frac{1-x}{(1+x)^3}, \quad |x| < 1$$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$\frac{(-3)(-4)(-5)\cdots}{(r!)(-3-r)(-4-r)(-5-r)\cdots}$

$(1-x)$

and find the general term.

$(-2-r)$

Evaluate $\sum\limits_{r=0}^{\infty} (-)^r \dfrac{(r+1)^2}{5^r}$.

$\frac{(2)(-3)(-4)(-5)\cdots}{(3-r+1)\ r!}$

$(-1)^{r}\frac{(+2+r)(+1+r)}{(1)(2)}$

4. abc is a triangle where $b\,(-4\tfrac{1}{2}, \tfrac{1}{2})$, $c\,(6, -1)$ and $x - y + 5 = 0$ is the equation of the line ab.

If the coordinates of the incentre of the $\Delta\, abc$ are $(0, k)$, where $k > 0$, find k and hence deduce the equation of the line ac.

Show that the distance between the incentre and the circumcentre of the $\Delta\, abc$ is greater than $2\tfrac{1}{4}$.

5. $(2, -1)$ and $(1, 1)$ are points on a circle S_1 and $x + 3y + 1 = 0$ is the equation of the tangent to the circle at $(2, -1)$. Find the equation of S_1.

S_2 is the circle $x^2 + y^2 - x + y - 2 = 0$ and L is the line containing $(2, -1)$ and $(1, 1)$. Prove that S_2 is the image of S_1 under the axial symmetry of the plane in L.
Write down the equation of a circle which is invariant under the transformation.

6. (i) Let $M = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$.

Sketch the set of points represented by the quadratic equation $(x\ \ y)\, M \begin{pmatrix} x \\ y \end{pmatrix} = 1$

and find the equations of the axes of symmetry of the set where these axes contain the origin.

(ii) Let $t \in \mathbf{R}$.

Find λ_1 and λ_2 for which $M \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix} = \lambda_1 \begin{pmatrix} t \\ \frac{t}{2} \end{pmatrix}$

and $M \begin{pmatrix} t \\ 2t \end{pmatrix} = \lambda_2 \begin{pmatrix} t \\ -2t \end{pmatrix}$

and sketch the lines $L_1 : x = t,\ y = \dfrac{t}{2}$

$L_2 : x = t,\ y = -2t$.

(iii) Let K be the unit circle $x^2 + y^2 = 1$.

Find (x_1, y_1) the coordinates of the point common to K and L_1 in the 1st quadrant.
Find (x_2, y_2) the coordinates of the point common to K and L_2 in the 2nd quadrant.

If $P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, evaluate $P^{-1} M P$.

7. (a) Show that $\frac{1}{2}(a - b)(\cos^2\theta - \sin^2\theta) + \frac{1}{2}(a + b) = a\cos^2\theta + b\sin^2\theta$.

(b) If $\tan 2\alpha = \frac{4}{3}$, find the values of $\tan\alpha$.

Find a local minimum and a local maximum of

$$\cos^2\theta - 2\sin 2\theta + 4\sin^2\theta.$$

8. Let the permutation p on the set $\{1, 2, 3\}$ be given by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Write out the permutations p^2 and p^3 where $p^2 = p \circ p$

Write out a permutation q on the set $\{1, 2, 3\}$, other than the identity, for which $q^{-1} = q$.

Show that the set $\{I, p, p^2, q\}$ is not closed under composition where

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, the identity permutation.

Let G be a permutation group under composition containing p and q as two of its elements.

Show that G is not commutative and find all its other elements.

Solve the equations

(i) $p^2 = q\,x$

(ii) $p^2 = x\,q$.

OR

8. Let $f'(a)$ and $f''(a)$ be the first and second derivatives, respectively, of $f(x) = x^4$ at $x = a$.

Show that the curve

$$y = f(3) + (x - 3)\,f'(3) + \frac{(x-3)^2}{2!}\,f''(3)$$

is a parabola, P, and find the equation of the tangent, T, to it at $x = 3$.

Verify that T is also the tangent at the point $(3, 81)$ to the curve K : $y = f(x)$.

If $(4, y_1)$, $(4, y_2)$, $(4, y_3)$ are points on T, P, K, respectively, verify that $y_1 < y_2 < y_3$.

Indicate, roughly, the shapes of K, P, T in the domain $3 \le x \le 4$.