LEAVING CERTIFICATE EXAMINATION, 1989

MATHEMATICS – HIGHER LEVEL – PAPER II (300 marks)

FRIDAY, 9 JUNE – MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown or if you have not indicated where a calculator has been used

1. (i) Differentiate the function

$$x \to \cos(3x^2 - \pi)$$

and find the value of the derivative at $x = \sqrt{\pi}$.

(ii) Express

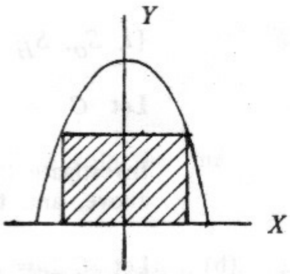
$$\frac{1 + i\tan\theta}{1 - i\tan\theta}$$

in the form $\cos k\theta + i \sin k\theta$, $k \in \mathbf{R}$.

(iii) Let $f(x) = \dfrac{x}{x+1}$ for $x \neq -1$.
If $k > 0$ and $h > 0$, show that

$$f(k+h) > f(h).$$

(iv) A rectangle has its base on the X axis and its other vertices on the curve $y = 6 - x^2$ as in the diagram.
Find the maximum and the minimum area of the rectangle.



(v) Show that for all x in the range $0 < x < 1$

$$\sin^{-1} x + \cos^{-1} x$$

is a constant.

(vi) Find the volume of the solid obtained by rotating the curve

$$y = \frac{1}{\sqrt{x}} + \sqrt{x}$$

about the X axis between the lines $x = 1$ and $x = 9$.

(vii) The sequence $u_1, u_2, u_3, \ldots$ is such that

$$u_1 = 1, \quad u_3 = 3, \quad u_{2n} = u_n$$
$$u_{4n+1} = 2u_{2n+1} - u_n$$
$$u_{4n+3} = 3u_{2n+1} - 2u_n$$

Evaluate u_7.

(viii) If $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive terms, show that

$$\sum_{n=1}^{\infty} \frac{u_n}{1+u_n}$$

is also a convergent series.

$\frac{u_n}{k_n} = \frac{\left(\frac{u_n}{1+u_n}\right)}{(u_n)} = \frac{1}{1+u_n}$

$\lim_{n\to\infty} \frac{u_n}{k_n} = \frac{1}{1+0} = 1$

(ix) Find the maximum value of x in the interval $0 < x < \frac{\pi}{2}$ for which

$$[1 + \cos^2 x + \cos^4 x + \ldots + \cos^{2(n-1)} x + \ldots] \geqslant 2.$$

(x) Find the length of the orthogonal projection of

$$2\vec{i} + 3\vec{j} \quad \text{on} \quad 12\vec{i} + 5\vec{j}.$$

OR

(x) n unbiased coins are tossed. Show that the probability of getting n heads or $(n-1)$ heads is

$$\frac{n+1}{2^n}.$$

2. (a) Let $z = x + iy$ where $x, y \in \mathbf{R}$.
Show on an Argand diagram in the z-plane the set H of z for which

$$|(z-1) + 4i| \leqslant 1.$$

On an Argand diagram in the w-plane $f(H)$, the image of H, under the transformation

$$w = f(z) = k(1-z), \quad k \in \mathbf{R}$$

is the circle $|w - i| \leqslant \frac{1}{4}$. Find the value of k.

(b) If $z^2 = \cos 2\alpha + i \sin 2\alpha$ where $i = \sqrt{-1}$
show that $z = \cos(n\pi + \alpha) + i \sin(n\pi + \alpha)$, for $n = 0$ or 1.

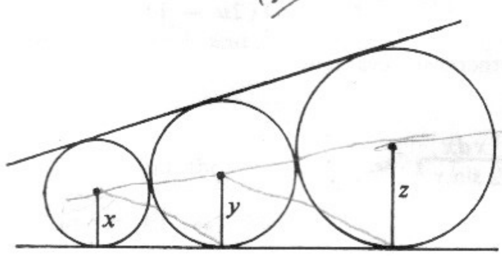
Let $z = \sqrt{2 - \sqrt{3}} - i\sqrt{2 + \sqrt{3}}$.

Express z^2 in the form $r(\cos\theta + i\sin\theta)$ and deduce that

$$\cos\frac{19\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

15×19

3.



$\frac{x+y}{y} = \frac{y+z}{z}$

$\frac{x}{y} + 1 = \frac{y}{z} + 1$

$\frac{x}{y} = \frac{y}{z}$

The diagram shows three circles touching two non-parallel lines and the middle circle touching the other two. The radii of the circles are of lengths x, y, z. Show, using similar triangles, or otherwise, that x, y, z are in geometric sequence.

4. (a) The derivative of

$$\log_e \frac{1 + \sin x}{1 - \sin x}$$

is $k \sec x$. Find the value of k.

(b) Find the value of the derivative of

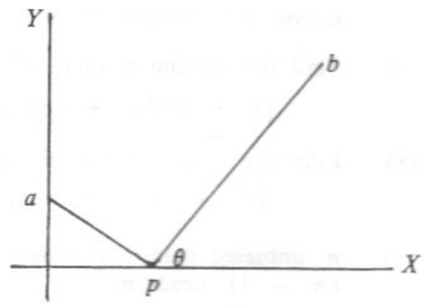
$$2^x \sin^{-1}(2x)$$

at $x = 0$.

(c) Find the equation of the tangent to the curve

$$x = \tfrac{1}{2}\left(1 - e^{-t}\right), \qquad y = 2t + e^t, \qquad t \in \mathbf{R}$$

at the point on the curve where $x = 0$.



5. $a(0, 1)$ and $b(4, 3)$ are points. A marble, p, rolls along the X axis in a positive sense away from the origin at a speed of 2·5 units per second.

At any instant the line bp makes an angle θ with the positive sense of the X axis. When p is at $(x, 0)$, $0 < x < 4$, show that $\tan\theta = \dfrac{3}{4 - x}$.

Find the rate at which θ, measured in radians, is increasing when the marble reaches the point $(3, 0)$.

Find also the rate of change of $|\angle apb|$ at that point.

6. (a) Evaluate $\displaystyle\int_0^1 x(1 + x^2)^5\,dx.$

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos 2x \sin 4x\,dx$

(c) Evaluate $\displaystyle\int_{-2}^{0} \frac{dx}{\sqrt{(5 + x)(1 - x)}}$

(d) Using long division, or otherwise, show that

$$\frac{u^2 - 1}{2u - 1} = \tfrac{1}{2}u + \tfrac{1}{4} - \frac{3}{4(2u - 1)}$$

and hence, or otherwise, evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\cos^3 x\,dx}{1 - 2\sin x}$$

7. (a) If

$$\frac{1}{(2n - 1)(2n + 1)} = \frac{a}{2n - 1} + \frac{b}{2n + 1}$$

for all $n \in \mathbf{N}$, find the value of a and the value of b and hence prove that

$$\sum_{n=1}^{\infty} \frac{1}{(2n - 1)(2n + 1)} = \tfrac{1}{2}.$$

(b) Verify that the sequence

$$0, \ \tfrac{3}{5}, \ \tfrac{8}{10}, \ \ldots, \ \frac{n^2 - 1}{n^2 + 1}, \ \ldots$$

converges to 1 and test for convergence the series

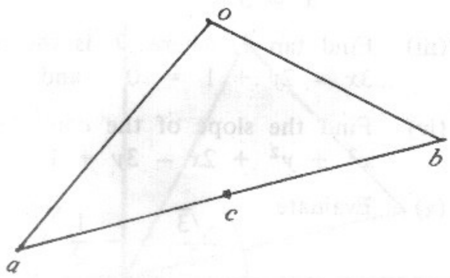
$$0 + \tfrac{3}{5} + \tfrac{8}{10} + \ldots + \frac{n^2 - 1}{n^2 + 1} + \ldots$$

$\lim_{n \to \infty} u_n = 1$

(c) Find the range of values of $x > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^2 + 1} x^n$$

converges.

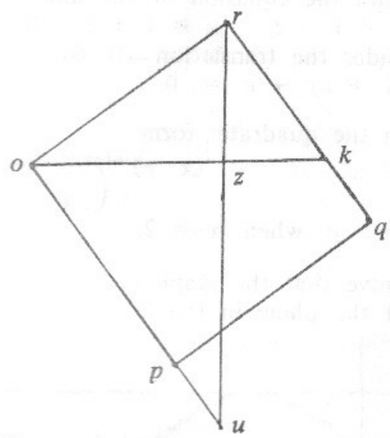


8. (a) In the Δoab,

c is the midpoint of $[ab]$.

Taking the point o as origin express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$ and hence, or otherwise, show that

$$|\vec{a}|^2 + |\vec{b}|^2 = 2\,|\vec{ac}|^2 + 2\,|\vec{c}|^2.$$



(b) $opqr$ is a square and k is a point such that

$$|rk| : |kq| = 2 : 1.$$

z is a point in $[ok]$ such that

$$|oz| : |zk| = 2 : 1, \text{ also.}$$

The line rz, produced, meets the line op, produced, at u.

Taking the point o as origin express $\vec{k}$ and $\vec{z}$ terms of $\vec{p}$ and $\vec{r}$.

If $\vec{ru} = t\vec{rz}$, $t \in \mathbf{R}$, express $\vec{u}$ in terms of $\vec{r}$ and $\vec{p}$ and t and hence find the value of t.

Deduce that $|\vec{pu}| = |\vec{kq}|$ and hence express the area of $kpuq$ in terms of $|\vec{r}|$.

OR

8. Use your Tables, page 36, to find the least value of k for which

$$\text{probability } (z \geqslant k) \leqslant 0{\cdot}05.$$

A person Q claims he can guess what another person H is thinking. To test this claim H thinks of a colour and Q guesses what the colour is. An experiment consists of repeating this 20 times and results in Q being correct 14 times.

State the null hypothesis for this experiment.

On the basis of your null hypothesis find the probability that Q can guess correctly at least 14 times out of 20.

Does the result of the experiment show sufficient evidence for Q to claim, at the 5% level of significance, that he can guess what H is thinking ?