LEAVING CERTIFICATE EXAMINATION, 1989

MATHEMATICS – HIGHER LEVEL – PAPER I (300 marks)

THURSDAY, 8 JUNE – MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown or if you have not indicated where a calculator has been used

1. (i) Find the distance between the two lines

$$2x - 6y + 5 = 0$$
$$x - 3y + 6 = 0$$

(ii) Find the equation of the tangent to the circle

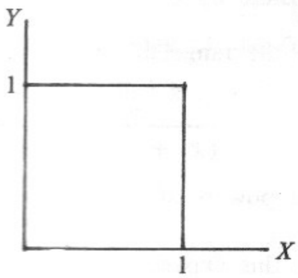
$$x^2 + y^2 + 2x - 6y + 5 = 0$$

at the point (1, 4).

(iii) Indicate the region of the unit square for which

$$|x - y| \leqslant \tfrac{1}{2}$$

and calculate the area of this region.



(iv) Find the number of arrangements of the letters of the word **CERTIFICATE** taking all the letters each time.

(v) Find the relation between the non-zero parameters μ and λ for which the line

$$\mu(3x - 2y + 3) + \lambda(x - 2y + 5) = 0$$

makes an angle measuring 45° with the positive sense of the X axis.

(vi) If

$$\begin{pmatrix} 1 & 1 \\ -1 & 3\cdot5 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

express $\begin{pmatrix} x \\ y \end{pmatrix}$ in the form $k\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $k \in \mathbf{R}$.

(vii) L is a line through the origin making an angle measuring 75° with the positive sense of the X axis.
Find the image of $\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}$ under S_L, the axial symmetry in L.

(viii) Show that the point $(\pi - x, y)$ is the image of the point (x, y) under the axial symmetry in the line $x = \frac{\pi}{2}$.
Deduce that the graph of $x \to \sin x$ is its own image under the axial symmetry in the line $x = \frac{\pi}{2}$.

(ix) Find the period of the function

$$x \to \sin 3\tfrac{1}{2}x \cos\tfrac{1}{2}x .$$

(x) K is the set of all subsets of $\{1, -1, i, -i\}$ and Δ is symmetric difference.
If

$$Y \,\Delta\, \{1, i\} \,\Delta\, Z = \Phi ,$$

express $Y \,\Delta\, Z$ as a subset of K.

OR (x) Find the locus of all possible vertices of the parabola

$$x^2 - 4x + 2ky = 0, \quad k \in \mathbf{R} .$$

Is there a value of k for which the parabola has its focus on the X axis ?

2. (a) If α, β, γ, are the roots of the equation

$$2x^3 - 3x^2 + 4x - 5 = 0 ,$$

verify that $\alpha^2 + \beta^2 + \gamma^2 = -\frac{7}{4}$

and evaluate $\alpha^3 + \beta^3 + \gamma^3$.

(b) Find the values of b for which the two equations

$$x^3 + x^2 - bx - b^2 = 0$$
$$x^3 + 5x^2 + 2bx - 2b^2 = 0$$

have a common root other than $x = 0$.

3. (a) (i) Show that

$$n^2 + 3n + 2$$

is even for all $n \in \mathbf{N}$.

(ii) Prove by induction that

$$n(n + 1)(n + 2), \quad n \in \mathbf{N}$$

is divisible by 6.

(b) Write down the range of x for which the expansion of

$$\frac{4x + 5}{(x + 2)^2 (x - 1)}$$

in ascending powers of x is valid.

Write down this expansion as far as the term containing x^2 .

4. Show that the three points $a(-1, 3)$, $b(5, -5)$, $c(8, -9)$ are collinear and verify that

$$|ab| : |bc| = 2 : 1.$$

A line through the point b cuts the line $y - 3 = 0$ in the first quadrant at k and also cuts the Y axis at h such that

$$|kb| : |bh| = |ab| : |bc|.$$

Find the slope of the line hk and find the ratio 2 : 1

$$\text{area of } \Delta\, abk : \text{area of } \Delta\, bhc .$$

5. Find the coordinates of the centre of the circle

$$2x^2 + 2y^2 - 6x + 2y - 5 = 0$$

and write down the length of its radius.

A line of slope $\frac{1}{3}$ is drawn through the centre of the circle and cuts the line

$$2x + 3 = 0$$

at the point p. Find the equations of the two tangents to the circle from p and verify that they are at right angles to each other.

6. (a) Find the matrix of the rotation of centre (0, 0) which maps the X axis onto the line $2x - y = 0$, and find the image of the line

$$2x - y + \sqrt{5} = 0$$

under this rotation.

(b) Find the components of the matrix M such that

$$B^{-1} \; M \; B \;=\; \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$$

where $B = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$.

If $\begin{pmatrix} x \\ y \end{pmatrix}$ represents a point of the plane, show that the line defined by

$$M \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix}$$

is perpendicular to the line defined by

$$M \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$$

7. (a) Express $3 \cos\theta + 4 \sin\theta$ in the form $a\cos(\alpha - \theta)$.
Hence write down the period and the range of the function

$$f : x \to 3\cos\tfrac{1}{5}x + 4\sin\tfrac{1}{5}x .$$

(b) If $\sec\theta - \tan\theta = x$, show that

$$\tan\tfrac{1}{2}\theta = \frac{1-x}{1+x} .$$

(c) Use your Tables, page 9, to express

$$\cos x + \cos 3x$$

as a product of two cosines and hence, or otherwise, solve

$$1 + \cos x + \cos 2x + \cos 3x = 0$$

for $0 \leqslant x \leqslant 2\pi$.

8. (a) Let

I be the identity transformation
S_O be the central symmetry of the plane in the origin
S_H be the axial symmetry of the plane in the line H, $y - x = 0$
S_K be the axial symmetry of the plane in the line K, $y + x = 0$

Express these four transformations as matrices and hence, or otherwise, fill in the Cayley table for the group

$\{I, S_O, S_H, S_K\}$ under composition.

Let $C = \{1, a, a^2, a^3\}$, where $a^4 = 1$.

Investigate if there is an isomorphism between the group of transformations, above, and the group C under multiplication.

(b) Let $G = \{1, a, b, a * b\}$ be a group under $*$

where 1 is the identity and where

$$a * a = 1$$
$$b * b = a.$$

Prove that $a * b = b * a$

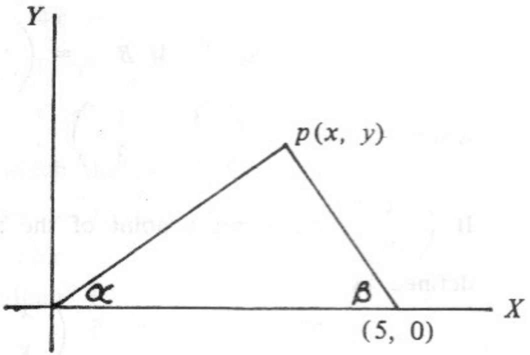
and evaluate $(a * b) * (a * b) * (a * b) * (a * b)$.

OR

8. $p(x, y)$ is a point such that

$\tan\alpha + \tan\beta = 4.$

Show that the locus of all possible p is a parabola.



Find the coordinates of the focus of the parabola and the equation of its directrix.

Let the parabola cut the X axis at h and k.
Verify that $\tan\alpha + \tan\beta = 4$ at these points.