LEAVING CERTIFICATE EXAMINATION, 1988

MATHEMATICS – HIGHER LEVEL – PAPER II (300 marks)

Attempt QUESTION 1 (100 marks) and FOUR other questions (50 marks each)

Marks may be lost if all your work is not clearly shown
or if you have not indicated where a calculator has been used

1. (i) If $y = \text{arc}\sec x$, show that

$$x\sqrt{x^2 - 1}\ \frac{dy}{dx} = 1.$$

(arc sec x is also written as $\sec^{-1} x$)

(ii) A rectangle is formed, as in diagram, having three vertices on the axes and one vertex on the line $2x + y = 4$. Find the maximum area of the rectangle.

(iii) Evaluate $\displaystyle\int_0^1 x(1 - x)^{99}\, dx$.

(iv) Express $\cos x - 1$ in terms of $\sin \frac{x}{2}$ and hence find

$$\lim_{x \to 0} \frac{3x^2}{\cos x - 1}.$$

(v) Evaluate $\left| \dfrac{z - 1}{1 - \bar{z}} \right|$ where $z \in C$.

(vi) Verify that $\dfrac{1}{(n + 1)(n + 2)} = \dfrac{1}{n + 1} - \dfrac{1}{n + 2}$

and hence evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n + 1)(n + 2)}$.

(vii) The sequence $u_0, u_1, u_2, \ldots, u_n, \ldots$ is such that

$$u_0 = -\tfrac{3}{2}$$
$$u_{n+1} = \tfrac{2}{3} u_{n-1}.$$

Verify for $n = 3$ that u_n can be written as $u_n = (\frac{2}{3})^{n-1} - 3$ and, assuming that this formula holds for all n, find the limit of the sequence.

(viii) Show that the curve $y = \dfrac{1}{1 + x}$ is its own image under the central symmetry of the plane in the point $(-1, 0)$.

(ix) Find the greatest value of x in the interval $0 < x < \frac{\pi}{2}$ for which

$$[1 + \sin^2 x + \sin^4 x + \ldots + \sin^{2(n-1)} x + \ldots] \leqslant 4.$$

(x) Five people are chosen at random. Show that the probability that at least two have their birthdays in the same month is greater than $\frac{3}{5}$.

OR

(x) Find two points on the line

$$\vec{r} = (4t - 3)\vec{i} + (8t - 2)\vec{j} \quad (t \in \mathbf{R})$$

and hence sketch the line.

Verify that the image of the line under the axial symmetry in the $\vec{j}$-axis is

$$\vec{r} = (3 - 4t)\vec{i} + (8t - 2)\vec{j} \quad (t \in \mathbf{R}).$$

2. (a) If $1, \omega_1, \omega_2$ are the roots of the equation

$$z^3 - 1 = 0$$

show that $\omega_2 = \omega_1^2$

and hence, or otherwise, show that

$$1 + \omega_1 + \omega_2 = 0.$$

(b) Let $z = x + iy$ where $x, y \in \mathbf{R}$.

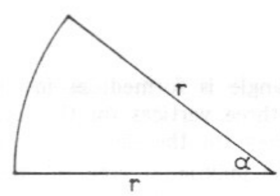
Indicate clearly on an Argand diagram in the z-plane the set K of z for which

$$1 \leqslant |1 + z| \leqslant 2.$$

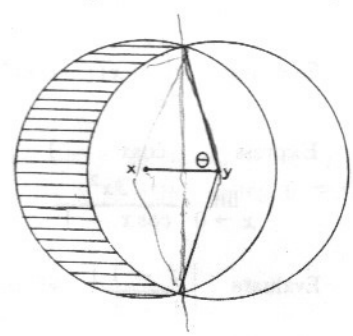
On an Argand diagram in the w-plane plot $f(K)$, the image of K under the transformation

$$w = f(z) = 1 - 3z.$$

3. The diagram shows a sector of a circle. Show that its area is $\frac{1}{2} r^2 \alpha$.



The diagram shows two circles having centres at x and y and having radii of unit length. Express the area of the hatched region in terms of θ.



4. (a) If $y = \left(\dfrac{x - 1}{x + 1}\right)^{\frac{1}{2}}$, show that $y\dfrac{dy}{dx} = \dfrac{1}{(x + 1)^2}$.

(b) (i) Given that

$$x = 4\cos\theta + 3\sin\theta$$

and

$$y = 3\cos\theta - 4\sin\theta$$

evaluate $\dfrac{dy}{dx}$ when $\theta = \dfrac{\pi}{2}$.

(ii) Differentiate $\sqrt{1 + x^2}\; e^{\sqrt{1+x^2}}$ with respect to x.

(c) The function f is defined by

$$f(x) = \log_e (1 + \sin x)$$

for $-\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Find the maximum value of $f(x)$ and the size of the angle which the tangent to its graph at $x = \pi$ makes with the positive sense of the Xaxis.

Write down the equations of its two asymptotes.

5. A vessel in the shape of a right circular cone has a height of 20 cm and the diameter of its base has a length of 20 cm. The vessel is placed so that it is standing on its vertex with its axis vertical. Water is flowing into the vessel at the rate of 100 cm^3 per minute. Find in terms of π the rate at which the height of the water is rising in the vessel when it is $\frac{1}{8}$ full.

6. Evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta - \int_0^{\frac{\pi}{4}} \sin^2 \theta \, d\theta$

(b) $\displaystyle\int_1^5 \frac{x\,dx}{x^4 + 10x^2 + 25}$

(c) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin^3 x \, dx}{1 + \cos 2x}$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} e^{1 + \log_e (\cos x)} \, dx$.

7. (a) Verify that $n! > 2^{n-1}$ for $n = 3$ and prove that $n! > 2^{n-1}$ for all $n \geqslant 3$ and $n \in \mathbf{N}$.

Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges and that its sum is less than 2.

(b) Find the range of values of $x > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(2n + 3)3^n}$$

converges.

(c) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{p^n}{p^n + q^n}, \quad p > 0, \; q > 0,$$

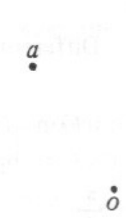
(i) when $p > q$

(ii) when $p < q$.

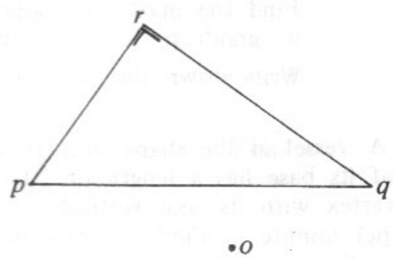
8. (a) a and b are points and o is the origin.
t is any point such that

$$\vec{a} \,.\, \vec{b} = \vec{t} \,.\, \vec{b}$$

Indicate by K on a diagram the set of all possible t.



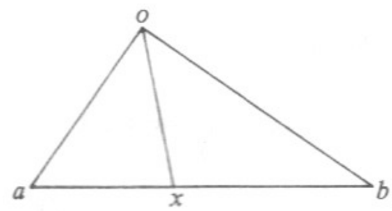
pqr is a right-angled triangle as in diagram.



Prove that

$$\vec{p} \,.\, \vec{q} - \vec{p} \,.\, \vec{r} = \vec{q} \,.\, \vec{r} - \vec{r} \,.\, \vec{r}.$$

(b) In the diagram the line ox bisects the $\angle aob$.



Show that $\vec{x}$ can be written as

$$\vec{x} = k\left[\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}\right] \text{ for } k \in \mathbf{R}.$$

If x divides $[ab]$ in the ratio $|ax| : |xb| = m : n$, write down an expression for $\vec{x}$ in terms of $\vec{a}, \vec{b}, m, n$.

Deduce that $|oa| : |ob| = |xa| : |xb|$.

OR

8. (a) Two dice, one coloured red and one coloured black, are thrown. Let

A be the event that the two numbers are the same
B the event that the combined score is $\geqslant 9$
C the event that at least one six is thrown.

(i) List the sample points in A, B, C.

(ii) Find $P(A)$, $P(B)$, $P(C)$, $P(A \cap C)$.

(iii) Verify that $P(A) + P(C) - P(A \cap C) = P(A \cup C)$.

(Note. $P(X)$ is the probability of the event X)

(b) The probability that an electric fuse, chosen at random from a production line, will function correctly is 0·98.

Find the probability that in a random sample of 1000 fuses at least 25 will be found to be defective.

If a random sample of 1000 was chosen each week from the production line and tested, estimate the percentage of weeks in which less than 25 fuses would be found to be faulty.