LEAVING CERTIFICATE EXAMINATION, 1988

MATHEMATICS – HIGHER LEVEL – PAPER I (300 marks)

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown
or if you have not indicated where a calculator has been used

1. (i) Investigate if the points (99, 35) and (0, 0) are on the same side of the line $5x - 13y - 37 = 0$.

(ii) Investigate if the lines represented by

$$x^2 - y^2 - x - y = 0$$

are perpendicular to each other.

~~$(x-y)(x+y+1)$~~

$(x+y)(x-y) - (x+y) = 0$

$(x+y)(x-y-1) = 0$

(iii) Solve

$$\log_4(6x+1) - 2 = 2\log_4 x .$$

$\log_4(6x+1) - \log_4(16) = \log_4 x^2$

$\frac{6x+1}{16} = x^2$

$16x^2 - 6x - 1$

$(8x+1)(2x$

$x = -\frac{1}{8}$ or $\frac{1}{2}$

(iv) Evaluate

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}^3$$

(v) If $a, b \in \mathbf{R}$ and $a > 0,\ b > 0$, prove

$$\left(\frac{1}{a} + \frac{1}{b}\right)\left(a + b\right) \geqslant 4 .$$

$1 + \frac{a}{b} + \frac{b}{a} + 1 \geq 4$

$\frac{a}{b} + \frac{b}{a} \geq 2$

$\frac{a^2+b^2}{ab} \geq 2$

a^2+b^2

a^2-2ab

$(a-b)^2$

True

(vi) Find the range of values of x for which

$$\frac{2x-1}{x-3} < \frac{2}{3} \quad \text{for } x \in \mathbf{R} \text{ and } x \neq 3 .$$

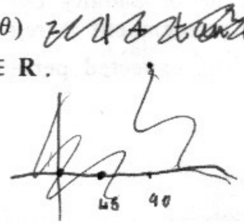
(vii) $\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$ represents a parallel projection of the plane onto a line A which contains the origin. Find the slope of A.

(viii) Sketch the graph of the function

$$\theta \to (1 + \tan^2\theta)(1 - \sin^2\theta)$$

~~$1 + \tan^2\theta - \sin^2\theta - \sin^2\tan^2\theta$~~

in the domain $0 \leqslant \theta < \frac{\pi}{2}$ for $\theta \in \mathbf{R}$.



$a^3+b^3=(a^2-ab+b^2)(a+b)$

$a^3-a^2b+ab^2$

$+a^2b-ab^2+b^3$

a^3+b^3

(ix) Investigate if there is a solution of
$$\cos^3 x + \sin^3 x = 0 \quad \text{for} \quad 0 \leqslant x \leqslant \tfrac{\pi}{2}.$$

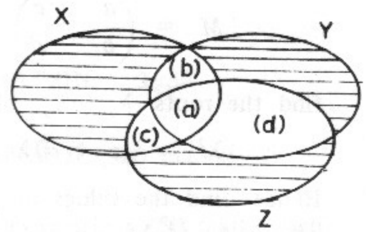
(x) $\{1, k, 11, 5\}$, mod 12, is a group under multiplication. Find k.

OR

(x) Write down the equation of the axis of symmetry of the parabola
$$x = 4t + 1, \quad y = 4t^2 + 1, \quad t \in \mathbf{R}.$$

2. (a) Find, correct to one decimal place, the least positive root of the equation
$$3(t-1)^3 + 16(t-1)^2 - 7(t-1) - 10 = 0.$$

(b) X, Y, Z are three sets and the hatching indicates some empty subsets. a, b, c, d indicate the number of elements in the respective subsets.
If $\#(X) = 2$, $\#(Y) = 3$ and $\#(Z) = 4$, prove that $a \neq 0$ and evaluate a, b, c, d.



3. (a) Show by induction that
$$7^{2n+1} + 1$$
is divisible by 8 for all $n \in \mathbf{N}$.

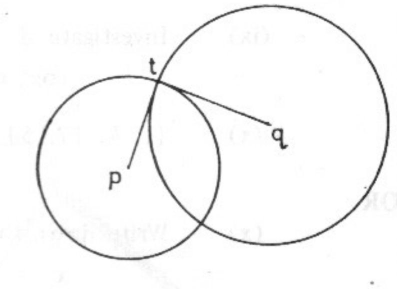
(b) Write out in ascending powers of x the first four terms of the expansion of
$$(1 - 2x)^{-2} \quad \text{for } |x| < \tfrac{1}{2}.$$
Find the value of x which makes the expansion equivalent to $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$ and hence evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$.

4. $L = 0$ and $K = 0$ are the equations of two intersecting lines. Prove that
$$\mu L + \lambda K = 0 \quad (\mu, \lambda \in \mathbf{R} \text{ where } \mu, \lambda \text{ are not both zero.})$$
is the equation of a line containing the point of intersection of $L = 0$ and $K = 0$.
Find the equations of the two lines which contain the point of intersection of the two lines
$$P: \ x - y + 7 = 0$$
$$Q: \ 4x + 5y - 3 = 0$$
and which cut the axes at equal distances from the origin.
One of these lines together with P and the Xaxis form a triangle T. Investigate if T has an axis of symmetry.

5. (a) $p(4, 2)$ is a point of the circle $x^2 + y^2 = 5x$. If $[pq]$ is a diameter of the circle, find the coordinates of q.
r and s are two points of the Xaxis such that
$$\text{area of } \Delta rpq = \text{area of } \Delta spq = \text{area of circle.}$$
Express $|rs|$ in terms of π.

(b) p and q are the centres of two circles

$$S_1: \quad x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$S_2: \quad x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$



which cut at t, as in diagram, and where $pt \perp qt$.

Prove that

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2 .$$

In particular let

$$S_1: \quad x^2 + y^2 + 2x - 3y - 4 = 0$$

and

$$S_2: \quad x^2 + y^2 + 2gx + 2fy + g = 0 .$$

If the radius of S_2 is of length 3, find all the values of g and f.

6. (a) Given that

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 16 & -2 \end{pmatrix}$$

find the roots λ_1, λ_2 of the equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0 .$$

Hence, find the values of k_1 and k_2 such that

$$M\begin{pmatrix} 1 \\ k_1 \end{pmatrix} = \lambda_1\begin{pmatrix} 1 \\ k_1 \end{pmatrix} \quad \text{and} \quad M\begin{pmatrix} 1 \\ k_2 \end{pmatrix} = \lambda_2\begin{pmatrix} 1 \\ k_2 \end{pmatrix}.$$

If $B = \begin{pmatrix} 1 & 1 \\ k_1 & k_2 \end{pmatrix}$, show that $B^{-1} MB = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

(b) Find the matrix of f, the rotation of the plane about the origin, which maps the X axis onto the line $3x - 4y = 0$. Find the equation of the image of the line $4x + 3y - 5 = 0$ under f.

7. (a) Solve

$$(1 - \tan\theta)(1 + \sin 2\theta) = (1 + \tan\theta)$$

for $0 \leqslant \theta \leqslant \pi$.

(b) Let f be the function

$$\theta \to 3 - 2\sin\tfrac{1}{2}\theta, \quad \theta \in \mathbf{R} .$$

Verify that 2π is not the period of f.

Use $\theta = 0$ in $f(\theta + l) = f(\theta)$ to find the period l, of f.

Calculate the maximum and minimum values of f in $0 \leqslant \theta \leqslant l$ and draw a rough sketch of f in the domain $0 \leqslant \theta \leqslant 10\pi$, using $\theta = 0$, π, 2π, 3π, ... 10π.

8. (a) $\{1, a, b, c\}$ is a group G under multiplication.

If $1 = a^2 = b^2 = c^2$, show that $ab \neq b$ and fill in the Cayley table for this group giving a reason for each entry.

Solve for x the equation

$$x^3 abc = b$$

(b) $\{1, i, -1, -i\}$ is a group C under multiplication where $i = \sqrt{-1}$.

Let f be an isomorphism which maps a subgroup of C onto a subgroup of G n (a) above.

Define all possible f.

Is there an f which maps $C \to G$? Explain your answer.

OR

8. Show that the equation of the parabola, S, which is the image of the parabola

$$y^2 = 4x$$

under the axial symmetry in the line

$$y = 2x$$

can be written in the form

$$(ax + by)^2 + k(bx - ay) = 0.$$

Find the equation of the axis of S and the equation of the tangent to S at its vertex.