LEAVING CERTIFICATE EXAMINATION, 1987

MATHEMATICS – HIGHER LEVEL – PAPER I (300 marks)

THURSDAY, 11 JUNE – MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

**Marks may be lost if all your work is not clearly shown or if you have not indicated where a calculator has been used**

1. (i) Solve $\log_{10} \dfrac{x^2 - 24}{x} = 1$ for $x \in \mathbf{R}$.

(ii) Find the range of values of $x \in \mathbf{R}$ for which

$$\frac{x-3}{x+1} < 2 \quad \text{when} \quad x + 1 < 0.$$

(iii) If $x = \dfrac{6t}{1+t^2}$ and $y = \dfrac{2(1-t^2)}{1+t^2}$, find the value of

$$\frac{x^2}{3^2} + \frac{y^2}{2^2}.$$

(iv) The code for a combination lock consists of two letters followed by three digits (e.g. *ZB* 020, *EE* 444). A part of the code contains the letter *B* and the digits 5 and 7. How many different permutations fit this description ?

(v) Find the equations of the lines represented by the equation

$$2x^2 + 5xy - 3y^2 + 7x + 14y + 5 = 0.$$

(vi) A line containing $p(5, 6)$ touches the circle

$$x^2 + y^2 - 4x - 4y + 4 = 0.$$

at $k$. Calculate $|pk|$.

(vii) Write

$$n\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1}$$

in the form $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^{-1}$

(viii) Find the matrix of the projection parallel to $y = -x + 1$ onto the line $x + 2y = 0$.

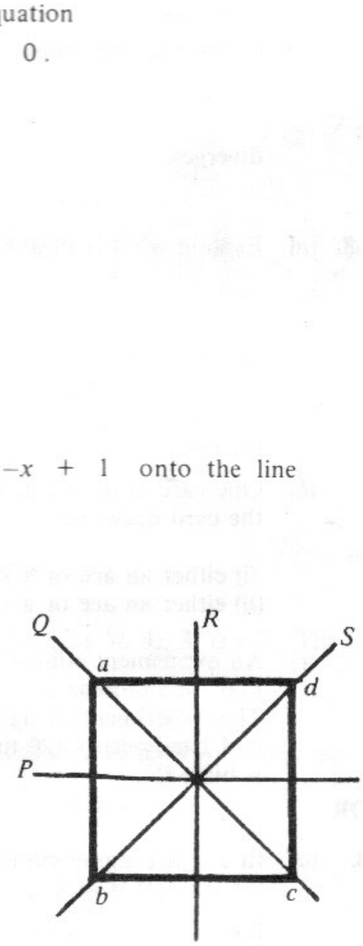
(ix) Find the period of the function

$$x \rightarrow 2\sin 3x \cos x.$$

(x) The square $abcd$ is mapped onto itself under each of the axial symmetries of the plane;

$$f_P, \; f_Q, \; f_R, \; f_S.$$

Investigate if $\{f_P, f_Q, f_R, f_S\}$ is a group under composition.




**OR** (x) Find the focus of the parabola

$$x = -8t - 2, \quad y = 4t^2 + 1, \quad t \in \mathbf{R}.$$

2. Show that the quadratic equation

$$(1 + a - b)x^2 + 2x + (1 - a + b) = 0$$

has real roots, one of which is independent of $a$ and $b$ and the other is not.

If this other root is $(-5)$, find the local minimum of the quadratic function

$$x \to (1 + a - b)x^2 + 2x + (1 - a + b), \quad x \in \mathbf{R}$$

and find also where the graph of the function intersects the $f(x)$ axis.

Draw a rough graph of the function.

3. (a) Show by induction that 17 divides

$$3^{4n+2} + 2.4^{3n+1} \quad \text{for } n \in \mathbf{N}.$$

$(1-x)^{-3}$

(b) Write down the first four terms of the expansion of

$$\frac{1 + x}{(1 - x)^3}, \quad |x| < 1$$

$(1+x)\left(1+3x+\frac{(-3)(-4)}{2!}(-x)^2+\frac{(-3)(-4)(-5)}{3!}(-x)^3\right)$

$4.3 = 12$

in ascending powers of $x$.

$1+3x-6x^2+10x^3$

$x+3x^2-6x^3$

$1+4x-3x^2+4x^3$

Write down the 20th term and the $n$th term and evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{2^{n-1}}.$$

$-\frac{39}{5}$

$S_n = \frac{1}{1}+\frac{4}{2}+\frac{9}{4}+\dots+\frac{n^2}{2^{n-1}}$

$\frac{1}{2}S_n = \frac{1}{2}+\frac{4}{4} \dots \frac{(n-1)^2}{2^{n-1}}$

4. (a) The lines

$$3x + 4y + 4 = 0 \quad \text{and} \quad 5x + 12y + 28 = 0$$

cut the $X$axis at $p$ and $q$, respectively. $k$ is a point in $[pq]$ which is equidistant from the two lines. Find this distance.

$2n-1$

(b) Find the equation of the line which contains the point of intersection of the two lines

$$4x - 4y + 3 = 0 \quad \text{and} \quad 4x + 4y - 3 = 0$$

and which is parallel to the line

$$12x + 4y + 6 = 0.$$

$\frac{1}{2}S_n = 1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}\dots\frac{2n-1}{2^{n-2}}$

$\frac{1}{4}S_n = \frac{1}{2}+\frac{3}{4}+\frac{5}{8}\dots\frac{2n-3}{2^{n-2}}$

Is there a line through the point of intersection of

$$4x - 4y + 3 = 0 \quad \text{and} \quad 4x + 4y - 3 = 0$$

which is not represented by the equation

$$4x - 4y + 3 + \lambda(4x + 4y - 3) = 0$$

$\frac{1}{4}S_n = 1+\frac{2}{2}+\frac{2}{4}+\frac{2}{8}$

for any value of $\lambda$? Give a reason for your answer.

$\frac{1}{2}S_n + \frac{n^2}{2^n} =$

$600 - 4(a)($

5. Write down the coordinates of the centre of the circle

$$S : y^2 = x(10 - x)$$

and find the length of its radius.

Prove that the line

$$K : 3x - 4y + 10 = 0$$

is a tangent to the circle and find the coordinates of the point of contact.

The line $K$ cuts the $X$axis at $p$ and makes an angle $\theta$ with the positive sense of the $X$axis. Let $f$ be the anticlockwise rotation of measure $2\theta$ about $p$.

Find the equation of $f(S)$.

6. (a) Let $f$ be the axial symmetry of the plane in the line $x - 3y = 0$. Write down the matrix of $f$.

Hence, or otherwise, find the equation of the image of the line $y - 5 = 0$ under $f$.

(b) Let $B = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ and $S = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$.

Evaluate $B^{-1}SB$ and write it in the form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

If every $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying $S\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$ is on a line $L$

and every $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfying $S\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} x \\ y \end{pmatrix}$ is on a line $M$,

prove $L \perp M$.

7. (a) (i) Show that $\dfrac{\sin\theta}{1 + \cos\theta} = \tan\frac{1}{2}\theta$.

(ii) If $\sin\theta = \dfrac{1 - x}{1 + x}$, express $\cos\theta$ in terms of $x$ and hence show that

$$\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sqrt{x}.$$

(b) Simplify the equation

$$\cos\left(x - \frac{\pi}{6}\right) - 3\sin\left(x + \frac{\pi}{3}\right) = 1$$

and hence find one value of $x$ which satisfies it.

8. (a) $P$ is the group $\{1, 5, 8, 12\}$ mod 13 under multiplication.

$Q$ is the group $\{1, 5, 7, 11\}$, mod 12 under multiplication.

Investigate whether there is an isomorphism $f : P \to Q$.

For each group write out $\{5^n \mid n \in \mathbf{N}\}$.

(b) The six functions

$f_1 : x \to x$ $\quad$ $f_2 : x \to \dfrac{1}{x}$ $\quad$ $f_3 : x \to 1 - x$

$f_4 : x \to \dfrac{1}{1 - x}$ $\quad$ $f_5 : x \to \dfrac{x}{x - 1}$ $\quad$ $f_6 : x \to \dfrac{x - 1}{x}$

form a group $G$ under composition.

(i) Find the least value of $k > 0$ for which
$$f_4^k = f_1$$
where $f^k$ means $f \circ f \circ f \ldots\ k$ times.

(ii) Verify that $\{f_1, f_4, f_6\}$ under composition is a subgroup of $G$.

**OR**

8. $p(at^2, 2at)$, $q\left(\dfrac{a}{t^2}, -\dfrac{2a}{t}\right)$ are two points on the parabola

$$y^2 = 4ax.$$

Show that the chord $[pq]$ is a focal chord.
(i.e. $[pq]$ contains the focus.)

Find the coordinates of the midpoint of $[pq]$ in terms of $a$ and $t$.

Deduce that the locus of the midpoints of the focal chords of $y^2 = 4ax$ is a parabola.

A focal chord of the parabola $y^2 = 8x$ has slope 1. Find its length.