LEAVING CERTIFICATE EXAMINATION, 1986

MATHEMATICS — HIGHER LEVEL — PAPER II (300 marks)

MONDAY, 16 JUNE — MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown or if you have not indicated where a calculator has been used

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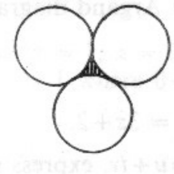
1. (i) Each circle in the diagram has radius r and each touches the other two. Find in terms of π and r the area of the shaded part enclosed by the three circles.



(ii) Express $\sqrt{2-2i\sqrt{3}}$ in the form $a+ib$ where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$.

(iii) Differentiate from first principles the function

$$x \to \frac{1}{1-x}$$

(iv) Show that if a sphere and a cube have the same surface area, the sphere has the greater volume.

(v) Let $u_n = 2+\frac{1}{n}$.
Investigate if the sequence

$$u_1, u_2, u_3, \ldots, u_n, \ldots$$

is monotonic decreasing.

(vi) The sequence

$$u_1, u_2, u_3, \ldots u_n, \ldots$$

is such that

$$u_1 = 1$$
$$u_{n+1} = \frac{u_n+2}{u_n+1} \quad \text{for } n \geqslant 1.$$

Write down the first three terms of the sequence.
If $\lim_{r\to\infty} u_r = k$, find k.

(vii) Find the coordinates of the point on the curve $y^2-6y-8x+9 = 0$ at which the tangent to the curve makes an angle of 45° with the positive sense of the x-axis.

(viii) Show that the asymptotes of the curve

$$y = \frac{1}{1-x}$$

intersect at (1, 0) and show that the curve is its own image under the central symmetry in (1, 0).

(ix) Evaluate $\lim_{x\to 0} \dfrac{2-2\cos^2 x}{\sin 2x}$

(x) From a pack of 52 playing cards a set of three cards is picked at random. What is the probability that all three cards are black?

OR

(x) Let $\theta < \pi$ be the measure of the angle between the two vectors

$$-3\vec{i}+4\vec{j} \text{ and } \vec{i}-\vec{j}.$$

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Find $\sin \theta$.

2. (*a*) Use de Moivre's theorem to show that

$$z_1 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} \quad \text{and} \quad z_2 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

are two of the roots of the equation $z^5 - 1 = 0$.
Show that $z_1.z_2$ is a root of the equation and investigate if

$$\frac{1}{z_1} \text{ and } \frac{1}{z_2}$$

are also roots.

(*b*) On an Argand diagram plot the set

$$K = \{z | 3x - 3y + 2 = 0\}, \quad z = x + iy.$$

Let $w = 3z + 2$.

If $w = u + iv$, express u and v in terms of x and y.

On an Argand diagram in the u, v-plane plot the image of K under the transformation $w = 3z + 2$.

$a^2+b^2-ab > ab$

$a^2+b^2 > 2ab$

$\frac{a^2+b^2}{2} > ab$

$a^4 - 2a^2b^2 + b^2 > 0$

$(a-b)^2 > 0$

$a^3 + b^3 \geq a^2b + b^2a$

3. a, b, c are three positive numbers, no two of which are equal.
Show that

$$a^3 + b^3 > a^2b + ab^2$$

and write down similar inequalities for $b^3 + c^3$ and $c^3 + a^3$.
Deduce that

$$3(a^3 + b^3 + c^3) > (a^2 + b^2 + c^2)(a + b + c).$$

$\frac{a^2}{b} + \frac{b^2}{a} > a + b$

$\frac{a^2}{b} - a + \frac{b^2}{a} - b > 0$

$a\left(\frac{a}{b} - 1\right) + b\left(\frac{b}{a} - 1\right) > 0$

4. (*a*) If $y = \dfrac{1}{1 + \cos x}$, show that

$$y^2 = \operatorname{cosec} x \frac{dy}{dx}.$$

(*b*) (i) Evaluate the derivative of

$$[\log(1 + \cos x)]^2$$

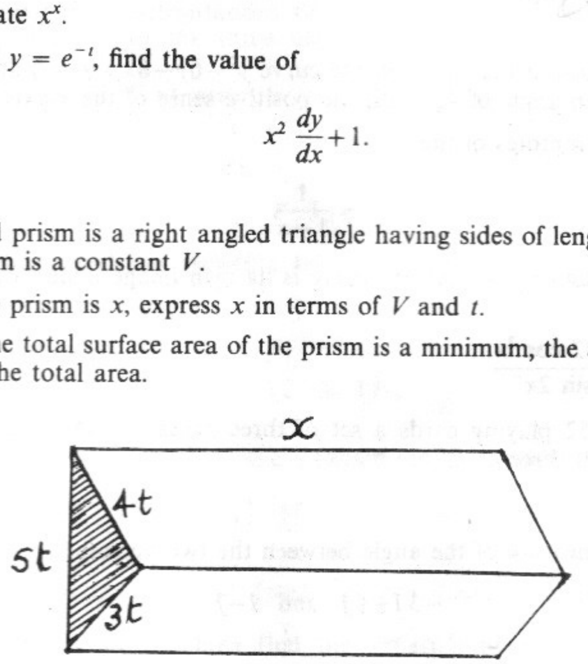
at $x = \dfrac{\pi}{2}$.

(ii) Differentiate x^x.

(*c*) If $x = e^t$ and $y = e^{-t}$, find the value of

$$x^2 \frac{dy}{dx} + 1.$$

5. The base of a solid prism is a right angled triangle having sides of lengths $5t$, $4t$, $3t$ ($t \in \mathbf{R}$). The volume of the prism is a constant V.
If the height of the prism is x, express x in terms of V and t.
Show that when the total surface area of the prism is a minimum, the area of the two triangular ends is one third the total area.



6. (*a*) Evaluate

$x < \frac{3}{4}$

$x^2 < \frac{9}{16}$

(i) $\displaystyle\int_0^1 \frac{x}{x+1}\,dx$ and $\displaystyle\int_1^2 \frac{x+1}{x}\,dx$

(ii) $\displaystyle\int_0^1 \frac{2x+1}{1+x^2}\,dx$

(iii) $\displaystyle\int_0^{\pi/4} \frac{dx}{\cos^2 x}$

(iv) $\displaystyle\int_0^1 \frac{dt}{1+e^{-t}}$

(*b*) The area between the curve $xy = 4$, the x-axis and the lines $x = 1$, $x = 2$ is the same as the area between the curve, the x-axis and the lines $x = t$, $x = 1$ where $0<t<1$. Find the value of t.

7. (*a*) Use the Comparison test to test for convergence the series

$$\sum_{n=1}^{\infty} \frac{4^n}{4^n+5^n}.$$

(*b*) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}$$

and find the range of values of $x>0$ for which the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$$

converges.

(*c*) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$$

diverges.

$a^3 + b^3$

$(a+b)(a^2 + b^2)$

$a^3+b^3 + a^2b + b^2a$

$a^3+b^3 + ab(a+b)$

8. (*a*) Explain what is meant by the axiom

$$P(S) = 1$$

where S is the sample space.
A die is loaded so that the probability of a particular number being thrown is proportional to that number so that $P(1) = k$, $P(2) = 2k$, $P(3) = 3k$ etc. Find the probability that 1 is thrown.

(*b*) One card is drawn at random from a pack containing 52 cards. Find the probability that the card drawn is

(i) either an ace or a king
(ii) either an ace or a red card.

$(a+b)(a^2 - ab + b^2)$

(*c*) An experiment consists of throwing an unbiased penny five times. What is the probability that the outcome is 4 heads and 1 tail?
This experiment, using a different penny, is performed 960 times and the outcome of 4 heads and 1 tail occurs 120 times. At the 5% level of significance can one conclude that this penny is biased?

OR

8. (*a*) In a $\triangle abc$ the circumcentre is at the origin. If $\vec{k} = \vec{a}+\vec{b}+\vec{c}$, prove that the orthocentre of the triangle is at k.

If $\vec{k} = -2\vec{i}+2\vec{j}$ and $\vec{a} = -2\vec{i}+6\vec{j}$, show that the line bc is parallel to the $\vec{i}$-axis and express $\vec{b}$ and $\vec{c}$ in terms of $\vec{i}$ and $\vec{j}$ where a, b, c are in anticlockwise order.

(*b*) If $\vec{u} = 4\vec{i}+2\vec{j}$, find two vectors $\vec{x}$ and $\vec{y}$ each of length $\sqrt{10}$ and such that ox and oy make equal angles of 45° on either side of ou, where o is the origin. Verify that $oxuy$ is a square.