LEAVING CERTIFICATE EXAMINATION, 1986

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

THURSDAY, 12 JUNE - MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

1. (i) If $p > 0$, $q > 0$ and $\dfrac{p}{q} = \sqrt{2}$, prove that $p - q < q$.

   (ii) Find the range of values of x for which
   $$|1 - x| < 5, \quad x \in \mathbf{R}.$$

   (iii) Find how many 5 digit numbers can be formed in which the first and last digits are greater than 5, the three centre digits are identical and the last digit is odd.

   (iv) Investigate if the point (30, 15) and the origin are on the same side of the line $10x - 20y = 1$.

   (v) The circles $x^2 + y^2 + 4x + 4y - 17 = 0$ and $x^2 + y^2 + 16x + 16y + 19 = 0$ intersect at p and q. Find the equation of the line pq.

   (vi) Find the coefficient of xy in the quadratic form
   $$(x \quad iy)\begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} x \\ -iy \end{pmatrix}$$
   where $i = \sqrt{-1}$.

   (vii) f is the axial symmetry of the plane in the line $\sqrt{3}\,y = x$. Write the matrix of f in the form $\frac{1}{2}M$.

   (viii) Let $\cos(4x - 3) = \cos[4(x + k) - 3]$ for all $x \in \mathbf{R}$. By choosing a suitable value for x, or otherwise, show that $\frac{\pi}{2}$ is the least positive value of k.
   (You may not assume that 2π is the period of the function $y \rightarrow \cos y$.)

   (ix) Show that the graph of the function $x \rightarrow \cos x\ (= y)$ is its own image under the axial symmetry in the y-axis.

   (x) The set $\left\{\dfrac{1 + 2m}{1 + 2n} \;\middle|\; m, n \in \mathbf{Z}\right\}$ is a group under multiplication.
   Find the relation between m and n that gives the identity element.

   OR

   (x) $2x - 2y + 5 = 0$ is a tangent to the parabola $y^2 = 4ax$.
   Find the coordinates of the point of tangency.

2. (a) If x, y, z are in arithmetic sequence, find the value of t which satisfies
   $$\begin{aligned} tx - 5y + 4z &= 3 \\ 2x + ty - 3z &= -5 \\ x - 2y + tz &= 0. \end{aligned}$$
   If $x - y + z = 17$, solve for x, y, z.

   (b) Show that the cubic equation
   $$x^3 - 3x^2 - 9x + 3 = 0$$
   has three real roots between -3 and 5.

   If the roots are written in order of magnitude, find the middle one correct to one place of decimals.

3. (a) For $|x| < k$, find the maximum value of k for which $(2 + x)^{\frac{1}{2}}$ has a binomial expansion.

Write out in ascending powers of x the first three terms in the expansion of $(2 + x)^{\frac{1}{2}}$ and use these terms to estimate $\sqrt{101}$.

(b) Write the expression

$$\frac{365(365 - 1)(365 - 2) \ldots (365 - 24)}{365^{25}}$$

in the form

$$1\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right) \ldots \left(1 - \frac{24}{x}\right)$$

and use the approximation

$$\log_e (1 - y) = -y$$

to estimate the value of the expression. (see Tables p. 31)

4. M and N are the two lines $3x + 4y - 3 = 0$ and $12x + 5y + 10 = 0$, respectively.

Find the equation of the bisector of that angle which contains the origin and verify that the bisector cuts the x-axis at the negative side of the origin.

M and N intersect at q and M cuts the x-axis at p. The bisector cuts the x-axis at b. Find the area of the Δpqb.

$k(x, y)$ is a point on the same side of the bisector as the origin. If

$$\text{area of } \Delta kqb = 2 \text{ area of } \Delta pqb,$$

find the relation between x and y.

Find the coordinates of k if it is on M.

5. Verify that

$$x(x - a) + y(y - b) = k(bx + ay - ab)$$

is the equation of a circle through the points $(a, 0)$ and $(0, b)$ for $k \in \mathbf{R}$.

Find the equation of the circle through the points $(1, 0)$ and $(0, 2)$ which has its centre on the line $x + 3y - 11 = 0$.

Verify that the origin is outside this circle and find the slopes of the tangents to the circle from the origin. (You may leave your answers in surd form.)

6. (a) Show that the axial symmetry of the plane in the $\vec{i}$-axis is a linear transformation.

If L is the line through the origin making an angle θ with the $\vec{i}$-axis, show that

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

is the matrix of the axial symmetry of the plane in L.

Plot the points $(\vec{i} + \sqrt{3}\vec{j})$, $2(\vec{i} + \sqrt{3}\vec{j})$, $3(\vec{i} + \sqrt{3}\vec{j})$ and hence show on a diagram the line $t(\vec{i} + \sqrt{3}\vec{j})$ for $t \in \mathbf{R}$.

Find the image of the vector $\begin{pmatrix} -2 \\ \sqrt{3} \end{pmatrix}$ under the axial symmetry in the line $t(\vec{i} + \sqrt{3}\vec{j})$, $t \in \mathbf{R}$.

(b) Let $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a > 0$. Let $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

If $M\vec{x} = \vec{x}$ and $|\vec{x}| = 1$, find $\vec{x}$.

Let $\vec{y} \perp \vec{x}$, $|\vec{y}| = 1$ and $\vec{y}$ in the first quadrant.

Find $\vec{y}$.

If $M\vec{y} = \mu\vec{y}$, find $\mu \in \mathbf{R}$.

7. (a) If $p(\cos A + \tan B \sin A) = \tan B$, show that

$$p = \frac{\sin B}{\cos(A - B)}.$$

(b) Express $\sin(A + B)$ in terms of sines and cosines.

Write $3\sin 3x + 4\cos 3x$ in the form $r\sin(3x + \theta)$ and hence write down the period and range of the function

$$x \to 3\sin 3x + 4\cos 3x.$$

Draw a rough graph of the function

$$x \to r\sin 3x$$

where $0 \leqslant x \leqslant 2\pi$ and $x \in \mathbf{R}$.

What transformation maps the graph of $x \to r\sin 3x$, $x \in \mathbf{R}$ to the graph of $x \to r\sin(3x + \theta)$, $x \in \mathbf{R}$.

Take $\theta = \frac{\pi}{3}$ and using the graph of $x \to r\sin 3x$ indicate the graph of $x \to r\sin(3x + \theta)$.

8. (a) Assuming that the multiplication of integers, mod 8, is associative, show that the set

$$A = \{1, 3, 5, 7\}, \mod 8$$

is a group under multiplication.

Show also that the set of functions

$$B = \{g_1, g_2, g_3, g_4\}$$

is a group under composition where

$$g_1(x) = x, \quad g_2(x) = -x, \quad g_3(x) = \frac{1}{x}, \quad g_4(x) = -\frac{1}{x}.$$

If f is an isomorphism which maps the group $A, \bullet \to B, \circ$ verify that

$$f(5 \bullet 7) = f(5) \circ f(7)$$

where "$\bullet$" means multiplication.

(b) Find the number of permutations of three objects taken all a time and write down all the permutations of the three digits 1, 2, 3 taken all at a time.

$G, \circ$ is the group of all permutations of the form $\begin{pmatrix} 1 & 2 & 3 \\ p & q & r \end{pmatrix}$ under composition where $1 \to p$, $2 \to q$, $3 \to r$.

Two elements of $G, \circ$ are $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.

Write out the remaining elements and say which is the inverse element of $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

Solve for t

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

OR

8. Show that $p(2, 4)$ is a point of the parabola $y^2 = 8x$.

If m is the slope of a chord $[pq]$ of the parabola, express the coordinates of q in terms of m.

Let $k(x, y)$ be the midpoint of $[pq]$. Write down a relation between y and m.

If q is any point on the parabola find the equation of the locus of k.