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LEAVING CERTIFICATE EXAMINATION, 1984

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

WEDNESDAY, 13 JUNE - MORNING, 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) If $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, evaluate $|1 + z|$, where $i = \sqrt{-1}$.

(ii) If $a, b, d \in \mathbf{R}$, show that the roots of the quadratic equation

$$x^2 - (a + d)x + (ad - b^2) = 0$$

are real.

(iii) Differentiate from first principles the function $x \to \sqrt{x}$ for $x > 0$.

(iv) If $q = 1 - p$, find the maximum value of pq.

(v) If $\int_0^t (e^x + 1)dx = e$,

write down a suitable value for t.

(vi) Test for convergence

$$\sum_{n=1}^{\infty} \frac{n - 1}{n^3 - 1} .$$

(vii) A sequence

$$u_1, u_2, u_3, \ldots, u_r, \ldots$$

is defined as follows:

$$u_1 = 0, \; u_2 = 1, \; u_{r+1} = \frac{u_r + \sqrt{u_r \cdot u_{r-1}}}{2} .$$

Find u_4 and express it in the form $a + b\sqrt{c}$, for $a, b, c \in \mathbf{Q}$.

(viii) Let $S_n = 1 + 2 + 3 + \ldots\ldots + n$. Write down a formula for S_n and and then use ΣS_n to evaluate

$$1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + 30).$$

(ix) Evaluate

$$\lim_{x \to 0} \frac{x^2}{(1 - 10x) - (1 - x)^{10}}$$

(x) Show that the standard deviation of

$$x_1, x_2, x_3, \ldots, x_n$$

is the same as the standard deviation of

$$x_1 + k, x_2 + k, x_3 + k, \ldots, x_n + k .$$

OR

(x) If $\vec{x} = \vec{i} - 2\vec{j}$, find one solution of $\vec{r}$ which satisfies the equation

$$\frac{\vec{r} \cdot \vec{x}}{|\vec{x}|^2} = \frac{1}{5} .$$

OVER →

2. (a) Prove that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for $n \in \mathbf{N}$ and $i = \sqrt{-1}$.

Deduce that $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$.

Solve

$$x^2 = 1 + i\sqrt{3}.$$

(b) Let $z = x + iy$ and $w = u + iv$ where $x, y, u, v \in \mathbf{R}$ and $i = \sqrt{-1}$.

If $w = 3z + 7i$, express u and v in the terms of x and y.

On an Argand diagram with axes x and iy plot the set K where

$$K = \{z \mid x = 1\}.$$

On another Argand diagram with axes u and iv plot the set of points which correspond to K under the transformation $z \to w$.

3. (a) Let $p > 0$ and $q > 0$ and $p \neq q$. Show that the geometric mean of p and q is less than the arithmetic mean of p and q.

(b) $u_1, u_2, \ldots, u_r, \ldots$ is a sequence of functions defined on $\mathbf{R}$ such that

$$u_{r+1}(x) = \frac{d}{dx} u_r(x)$$

and $$u_1(x) = \sin x$$

Evaluate $\sum_{r=1}^{43} u_r(x)$ when $x = \frac{\pi}{3}$.

(c) Find a number $k \in \mathbf{N}$ such that for all $n > k$

$$\frac{n^2+1}{n^2} - 1 < 0{\cdot}001.$$

4. (a) (i) If $y = \frac{1}{x^2}\log_e x$, find the value of

$$\frac{dy}{dx} \text{ at } x = \sqrt{e}.$$

(ii) If $y = xe^{-x}$, show that

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

(b) Find the equation of the tangent to the curve

$$x = t - 2\cos t, \quad y = 2\sin t - \cos 2t \quad \text{for } t \in \mathbf{R}$$

at the point on the curve where $t = 0$.

(c) Make a copy in your answer-book of fig. (a) and fig. (b) keeping the position of the axes as shown.

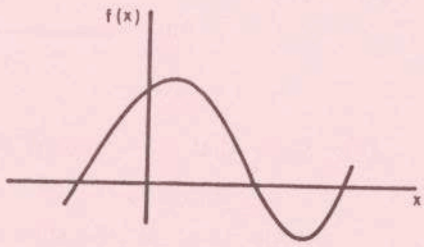


fig. (a)

fig (a) shows the graph of a function $x \to f(x)$. Using the axes in fig. (b) make a rough graph of the derived function $x \to f'(x)$, where $f'(x)$ means the derivative of x.

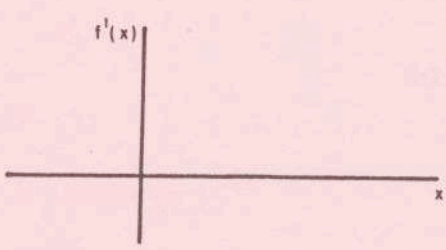
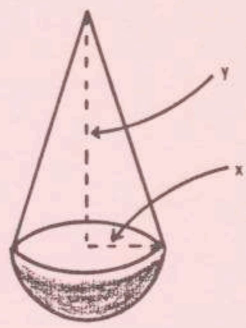


fig. (b)

5. A vessel consists of a cone standing on a hemisphere. The height of the cone is y cm and the radius of its base is x cm.



If the volume of the hemisphere is equal to the square of the volume of the cone, show that

$$xy^2 = \frac{6}{\pi} .$$

When the radius of the hemisphere is increasing at the rate of 2 cm/s, find the rate of change of the height of the cone when this height is 1 cm and investigate if the volume of the vessel at the same instant is either increasing or decreasing.

6. (a) Evaluate

(i) $\displaystyle\int_1^2 \frac{x^4 + 2x^2 + 1}{x^2}\, dx$

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$

(iii) $\displaystyle\int_0^1 x e^{-x^2}\, dx$.

(b) The graph of the function

$$x \to (x + 1)\sqrt{3 - x} , \quad -1 \leqslant x \leqslant 3$$

is rotated about the x-axis. Find in terms of π the volume generated.

7. (a) Test for convergence

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n + 1}}{n}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{(n + 1)\sin x} , \quad 0 < x < \frac{\pi}{2}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n + 1}}$

(b) Let $f(x) = x^2 + x + 2$ and let $0 < k < 1$.

Prove that

$$f(x) - 4 < k$$

when

$$0 < x - 1 < \frac{k}{10} .$$

OVER →

8. (a) (i) Prove that

$$P(E \setminus F) = P(E) - P(E \cap F),$$

where $P(X)$ means the probability of the event X and where the singletons are equally likely.

(ii) Two dice are thrown. Calculate the probability that the sum of the outcomes is even when 5 on either die is excluded when forming an even sum.

(b) A standard medication gives relief in 80% of cases. A new medication is tested on a population of 400 patients needing it and it gives relief to k of them. Find the least value of k which would indicate at the 5% level of significance that the new medication is better than the old.

OR

8. (a) Assuming that scalar product is commutative and distributive, prove that

$$|\vec{x} + \vec{y}|^2 - |\vec{x}|^2 - |\vec{y}|^2 = 2\vec{x} \cdot \vec{y}$$

and deduce that in a triangle abc

$$c^2 = a^2 + b^2 - 2ab \cos C$$

where a, b, c and C have their usual meanings (see Tables P. 9)

(b) Verify that the origin is the circumcentre of the triangle abc where

$$\vec{a} = 3\vec{i} + 4\vec{j}, \quad \vec{b} = -4\vec{i} + 3\vec{j}, \quad \vec{c} = -5\vec{i}$$

and express the orthocentre $\vec{h}$ in terms of $\vec{i}$ and $\vec{j}$.

If $\vec{h} + \alpha\overrightarrow{ha}$ is a vector on the $\vec{j}$-axis, find the value of $\alpha \in \mathbf{R}$ and verify that the point representing this vector is on the circumcircle.