LEAVING CERTIFICATE EXAMINATION, 1984

MATHEMATICS - HIGHER LEVEL - PAPER I (300 marks)

FRIDAY, 8 JUNE - MORNING 9.45 to 12.15

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) Find the one value of $x \in \mathbf{R}$ and the one value of $y \in \mathbf{R}$ which satisfies

$$(2x - 1)^2 + (2y + 1)^2 = 0 .$$

(ii) Prove $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

(iii) Four distinguishable dice are thrown once. Calculate the number of different ways in which only one six is thrown.

(3, 2, 4, 6 or 5, 6, 5, 5, for example, are possible outcomes).

(iv) Calculate the perpendicular distance between the parallel lines

$$3x - 4y = -10 \text{ and } 3x - 4y = 15 .$$

(v) Verify that the line containing the points of intersection of the two circles

$$x^2 + y^2 + 2x - 4y + 1 = 0 \quad \text{and} \quad x^2 + y^2 - 8x - 9 = 0$$

is perpendicular to the line containing their centres.

(vi) Find the coefficient of xy in the quadratic form

$$(x \;\; y)\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} .$$

(vii) Find the matrix of the rotation which maps

$$\begin{pmatrix} 6 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

(viii) Draw a rough graph of the function

$$x \to \tan^{-1} x = f(x), \quad x \in \mathbf{R}$$

where $\tan^{-1} x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Is there a point x on the x-axis for which $f(x) = 2 \tan^{-1} 2$? Give a reason for your answer.

(ix) Find $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{\sin 3x}$.

(x) Let $p = \begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix}$ and $q = \begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}$.

Write the composite map $p \circ q$ in the form $\begin{pmatrix} a & b & c & d \\ ? & ? & ? & ? \end{pmatrix}$.

If a, b, c, d are the vertices of a square, name the transformation $p \circ q$.

OR

(x) $t\ (x, y)$ is any point such that its distance from the line $x = 2$ is equal to its distance from the point $(3, -2)$. Find the equation of the locus of t.

2. (a) Let α, β, γ be the roots of the equation

$$x^3 - px + q = 0.$$

(i) Find $\alpha^2 + \beta^2 + \gamma^2$.

(ii) Find the equation whose roots are

$$1 - \alpha, \quad 1 - \beta, \quad 1 - \gamma.$$

$(\alpha+\beta+\gamma)(\alpha+\beta+\gamma)$
$(\alpha^2+\alpha\beta+\alpha\gamma)$
$+(\alpha\beta+\beta^2+\beta\gamma)$
$(\alpha\gamma+\beta\gamma+\gamma^2)$

(b) Express

$$3x + 3y - 4z + 2$$

as a linear combination of

$$x - y + 2z - 4 \quad \text{and} \quad 2x + y - z - 1.$$

Find one solution of the simultaneous equations

$$\begin{aligned} x - y + 2z &= 4 \\ 2x + y - z &= 1 \\ 3x + 3y - 4z &= -2 \end{aligned}$$

and say, giving a reason, whether this solution is unique.

3. (a) Evaluate $\begin{pmatrix} -\frac{1}{3} \\ 3 \end{pmatrix}$.

Expand

$$(1 - 3x)^{-\frac{1}{3}}$$

in ascending powers of x to four terms and use your expansion to evaluate

$$\sqrt[3]{\frac{100}{97}}$$

correct to four decimal places.

(b) You are given that the total number of ways of placing r identical coins into n boxes is

$$\binom{n + r - 1}{r}$$

(For example all the coins could go into just one box, the other boxes remaining empty.)

Use this result to calculate the total number of possible outcomes when 4 identical dice are thrown.

4. $x^2 - y^2 - 8x - 6y + 7 = 0$ is the equation of a pair of lines. One of the lines is $x + y - 1 = 0$. Find the equation of the other.

K is a parallelogram having the above two lines as diagonals and $x - 5y = 7$ is the equation of a line containing one side of K. Find the coordinates of the four vertices of K and calculate its area.

Investigate if K is its own image under the composition of axial symmetries $S_1 \circ S_2$, where S_1 is the axial symmetry in one diagonal and S_2 is the axial symmetry in the other.

5. C is the circle $x^2 + y^2 - 8x + 4y - 5 = 0$.
A circle K touches C internally and passes through r, the centre of C. If $3x - 4y + 5 = 0$ is the tangent common to both circles, find the equation of K.

H is the image of K under the central symmetry in r.
Find the equation of H and of the common tangent to H and C.

6. (a) Let $Q = \begin{pmatrix} 5 & 6 \\ 6 & 6 \end{pmatrix}$, $P = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Verify that $\vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ satisfies $Q\vec{x} - 2P\vec{x} = \vec{0}$ and find one non-zero vector $\vec{y}$ which satisfies $Q\vec{y} + 3P\vec{y} = \vec{0}$.

Using the values you found for x_1, x_2, y_1, y_2 evaluate

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

(b) Write down the matrix for the axial symmetry in a line through the origin which makes an angle A with the x-axis. Write down also the matrix of the central symmetry in the origin.
By finding the matrix for each axial symmetry, show that the axial symmetry in the line $2x + y = 0$ after the axial symmetry in the line $x - 2y = 0$ is equal to the central symmetry in the origin. (See Tables P. 9)

7. (a) Find the angles A in the range $0 \leqslant A \leqslant 2\pi$ which satisfy

$$\tan A + \cos A = \sin A + 1.$$

$\frac{\sin a}{\cos a} + \cos a = \sin A + 1$

tan

(b) Write the expression

$$\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1}$$

$\cos a - \sin a - 1 + \frac{\sin a}{\cos a} = 0$

in terms of $\sin(\theta/2)$ and $\cos(\theta/2)$ and hence, or otherwise, show that the expression can be written as

$$\sec\theta\,(1 + \sin\theta).$$

$\cos^2 a - \sin a \cos a - \cos a + \sin a = 0$

$\cos a(\cos a - \sin a - 1) + \sin a = 0$

8. Assuming that the multiplication of matrices is associative, show that the set

$$D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

is a group under multiplication.

If $t \in D$, solve the equation

$$t\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{151} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{150}.$$

State, giving a reason, whether or not each of the following sets under multiplication is a subgroup of $D, \times$:

$$E = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$F = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Investigate if there is an isomorphism between $D, \times$ and Z_4 $+$, where Z_4 is the set of residue classes modulo 4.

OR

8. The point $(-2, 3)$ is the vertex of a parabola and $(-2, 2\frac{1}{4})$ is the focus.

Write down the equation of the parabola and draw a rough sketch of it.

Tangents are drawn to the parabola at p and q, the two points where the parabola is cut by the x-axis. Show that these tangents intersect at a point which is a distance $|pq|$ from the x-axis.

These two tangents and the x-axis form a triangle. Find the radius of the largest circle with centre at $(-2, 3)$ which will fit into this triangle.