LEAVING CERTIFICATE EXAMINATION, 1983

MATHEMATICS – HIGHER LEVEL – PAPER I (300 marks)

FRIDAY, 10 JUNE – MORNING 9.30 to 12.00

Attempt **QUESTION 1** (100 marks) and **FOUR** other questions (50 marks each)

Marks may be lost if all your work is not clearly shown

1. (i) $3x + 5 = p(x + 1) + q(x - 1)$ for all values of x. Find p and q.

(ii) Evaluate the fourth term in the expansion of

$$(1 - x)^{-\frac{1}{2}}$$

when $x = \frac{1}{5}$ and write your answer in the form k^2.

(iii) How many different arrangements can be made from the letters of the word MATHEMATICS taking all the letters each time?
(You may leave your answer in factorial form)

(iv) Write the equation of the line

$$x = \frac{2t - 1}{t - 1}, \qquad y = \frac{2t}{t - 1}$$

in the form $ax + by + c = 0$.

(v) The circles

$$(x + 2)^2 + (y - 1)^2 = 8$$
$$(x - 5)^2 + (y + 2)^2 = 50$$

intersect in two points. Using the Theorem of Pythagoras, or otherwise, prove that the tangents to both circles at a point of intersection are at right angles.

(vi) Write

$$(x \quad y) \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

in the form $ax^2 + bxy + cy^2$.

(vii) Find the matrix of the rotation about the origin which maps

$$\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \text{ to } \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

(viii) Prove $\dfrac{2 \tan A}{1 + \tan^2 A} = \sin 2A$.

(ix) Find the period and range of the function

$$f : \mathbf{R} \to \mathbf{R} : x \to \cos 3x \sin 3x.$$

(x) Let k be the map $\begin{pmatrix} x & y & z \\ z & y & x \end{pmatrix}$ which means that $x \to z,\ y \to y,\ z \to x$.

Write $k \circ k$ in the form $\begin{pmatrix} x & y & z \\ ? & ? & ? \end{pmatrix}$.

If x, y, z are the vertices of an isosceles triangle, name the transformation k.

OR

(x) Find the equation of a tangent to the parabola $x^2 = 2y$ from the point $(4,\ 3\frac{1}{2})$.

2. (a) Solve

$$\begin{aligned} 2x + \ \ y + \ \ z + 7 &= 0 \\ x + 2y + \ \ z + 8 &= 0 \\ x + \ \ y + 2z + 9 &= 0. \end{aligned}$$

(b) Show that the equation

$$x^3 - x + 2 = 0$$

has no positive root and find, correct to one place of decimals, its only negative root.

3. (a) If x^3 and higher powers of x may be neglected, find an approximation in the form $a + bx + cx^2$, for the expression

$$\frac{(1 + \frac{1}{2}x)^3}{\sqrt{1-x}}$$

and hence estimate the value of the expression when $x = \dfrac{8}{15}$.

(b) Writing $x + y + z$ as $(x + y) + z$, find the coefficient of x^3y^2z in the expansion of $(x + y + z)^6$

and show that this coefficient is equal to

$$\frac{6!}{3!\,2!\,1!}.$$

4. (a) Prove that for any $\mu, \lambda \in \mathbf{R}$

$$\mu(3x - 4y + 5) + \lambda(2x + y - 6) = 0$$

is the equation of a line containing the point of intersection of the two lines

$$\begin{aligned} 3x - 4y + 5 &= 0 \\ 2x + \ \ y - 6 &= 0. \end{aligned}$$

(b) $o(0,0)$, $a(2,4)$, $b(6, -2)$ are the vertices of a triangle. $p(x,y)$ is a point such that its distances from the two lines oa, ob are inversely proportional to the lengths $|oa|$, $|ob|$. Find the equation of the locus of p. Verify that this locus is a median of the triangle.

(Note: $g : h$ is inversely proportional to $m : n$ if $g : h = n : m$)

5. Find the coordinates of p, the point of contact of the tangent

$$3x - 4y + 13 = 0$$

to the circle

$$x^2 + y^2 + 6y - 16 = 0.$$

Write down the distance from the centre of the circle to a chord of length 6 of the circle and hence, or otherwise, find the slopes of the two lines through p from which chords of length 6 are cut off by the circle.

6. (a) L is a line through the origin which makes an angle of 30° with the $\vec{i}$-axis and f is the projection of the plane on L in the direction perpendicular to L. By considering the images under f of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ or otherwise, find the matrix of f.

(b) Let $Q = \begin{pmatrix} 5 & 6 \\ 6 & 6 \end{pmatrix}$ and $P = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$.

If $\vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find the values of $\lambda, \mu \in \mathbf{R}$ for which

$$Q\vec{x} = \lambda P\vec{x} \quad \text{and} \quad Q\vec{y} = \mu P\vec{y}.$$

Evaluate

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} Q \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}.$$

7. Use a right angled triangle to show that

$$\sin^2 x + \cos^2 x = 1$$

and hence, or otherwise, find the two values of x in the range $0 \leqslant x \leqslant 2\pi$ for which

$$\sin x - \cos x = 1.$$

Let $f : \mathbf{R} \to \mathbf{R} : x \to \sin x - \cos x, \quad 0 \leqslant x \leqslant 2\pi.$
Find the local maximum and local minimum of f and use the one set of axes to draw rough graphs of

$$x \to \sin x$$
$$x \to \cos x$$
$$x \to \sin x - \cos x$$

in the domain $0 \leqslant x \leqslant 2\pi$.

8. (*a*) The rotations about the origin of angles 90°, 180°, 270°, 360° which map a square onto itself form a group under composition. This group is represented by the following Cayley table:

o	R_k	R_t	R_u	R_p
R_k	R_p	R_u	R_k	R_t
R_t	R_u	R_p	R_t	R_k
R_u	R_k	R_t	R_u	R_p
R_p	R_t	R_k	R_p	R_u

(i) Which element represents $R_{360°}$?

(ii) Which two elements are their own inverses?

(iii) Name the angles represented by k, t, u, p respectively.

(iv) Write down the elements of the subgroup of order 2.

(*b*) A rectangle, not a square, is mapped onto itself under each of four transformations – two axial symmetries and two rotations These four transformations under composition are a group. Test if this group is isomorphic to the group in (*a*) above.

OR

8. The tangent at the vertex of a parabola is

$$x - y + 3 = 0$$

and the focus is at $(-2, 4)$.

Find the coordinates of the vertex and the equation of the directrix and hence find the equation of the parabola. Verify that the parabola cuts the y-axis at two points p and q, say, but does not cut the x-axis. Calculate $|pq|$ and draw a rough sketch of the curve.