LEAVING CERTIFICATE EXAMINATION, 1982

MATHEMATICS - HIGHER LEVEL - PAPER II (300 marks)

1. (i) Find the values of q for which

$$x^2 + 3x + 3 = q(x^2 + 5)$$

has equal roots.

(ii) If $z_1 = 4 + 3i$ and $z_2 = 3 + 2i$, where $i = \sqrt{-1}$, verify that

$$rl\ z_1\ \bar{z}_2 \leqslant |z_1|\ .\ |z_2|$$

(iii) Find the sum of the first 31 terms of the series

$$\frac{1 + 1^3}{1 + 1} + \frac{1 + 2^3}{1 + 2} + \frac{1 + 3^3}{1 + 3} + \ldots\ldots + \frac{1 + n^3}{1 + n}$$

(Hint: Factorise $1 + n^3$)

(iv) Prove by induction or otherwise that

$$\sum_{t=1}^{n} \frac{1}{t(t+1)} = \frac{n}{n+1}\ .$$

(v) Differentiate the function $x \longrightarrow \frac{1}{x}$ from first principles.

(vi) A tangent is drawn at the point (1, 2) to the curve

$$xy^2 + y = 6.$$

Find the slope of this tangent.

(vii) The sequence

$$u_1,\ u_2,\ u_3,\ \ldots\ldots,\ u_r,\ \ldots\ldots$$

is defined as follows:

$$u_1 = u_2 = 1$$

$$u_{4r} = 0 \quad \text{for all } r \in \mathbb{N}$$

$$u_{r+1} = u_r - 2u_{r-1} \quad \text{for all other } r.$$

Find u_{10}.

(viii) A sequence is said to be monotonic decreasing if $T_{n+1} < T_n$ for all n.

The nth term of a sequence is given by

$$T_n = \frac{n}{n+1}\ .$$

Express $T_{n+1} - T_n$ in terms of n and hence investigate if the sequence is monotonic decreasing.

(ix) The curve $y^2 = 2x$ is rotated about the x-axis between $x = 1$ and $x = 4$. Calculate the volume generated.

(x) Two factories, A and B, supply 60% and 40%, respectively, of the camera flash-bulb market, 5% and $7\frac{1}{2}$% of their respective products being defective. Calculate the probability that a bulb bought at random is defective.

OR

(x) Express in terms of $\vec{i}$ and $\vec{j}$ the centroid of all the triangles which have one vertex at $2\vec{i} - 5\vec{j}$ and midpoint of opposite side at $2\vec{i} + \vec{j}$.

2. (a) Let $z = x + iy$ and $\bar{z} = x - iy$, where $i = \sqrt{-1}$.

Show on the Argand diagram the locus, K, of z such that

$$|z - 1 - 4i| = |z - 5|.$$

Let f be the transformation defined by

$$f : z \rightarrow \tfrac{1}{2}(z + \bar{z}) + \tfrac{i}{2}(z - \bar{z}).$$

Show on the Argand diagram the set $f(K)$, the image of K under f.

(b) Using real numbers find the two quadratic factors of

$$x^4 + 1$$

and hence, or otherwise, write down the complex roots of

$$x^4 + 1 = 0.$$

(Hint: Let $x^4 + 1 = (x^2 + px + 1)(x^2 + qx + 1)$)

3. (a) Use the identity

$$n^4 - (n - 1)^4 = 4n^3 - 6n^2 + 4n - 1,$$

or otherwise, to show that

$$\sum_{t=1}^{n} t^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(b) Write out the first six terms of the expansion of

$$\frac{1}{1 + t}$$

in ascending powers of t and then use

$$\int_{o}^{x} \frac{dt}{1 + t}$$

to write out the first six terms of the expansion of

$$\log_e(1 + x)$$

in ascending powers of x.

Deduce the value of $\log_e 11$ correct to four places of decimals.

4. (a) (i) Show that the derivative of a constant is zero.

(ii) Evaluate at $x = \frac{1}{\pi}$ the derivative of

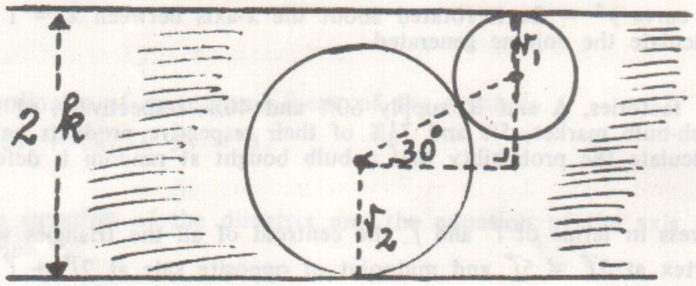
$$x \sin \frac{1}{x}.$$

(iii) If $y = \log_e \dfrac{1 + \sin x}{1 - \sin x}$, express $\dfrac{dy}{dx}$ in the form $p \sec q$.

(b) Let $y = e^{\sqrt{2x}} + e^{-\sqrt{2x}}$.

Find k such that

$$\sqrt{2x}\frac{dy}{dx} + y = ke^{\sqrt{2x}}.$$

5. 

From a strip of tin of width $2k$ two discs are cut out. One disc touches the top edge of the strip, the second disc touches the bottom edge and both discs touch each other, as in diagram. If the line joining the centres of the discs remains at the angle of 30° to the edge of the strip, show that

$$3(r_1 + r_2) = 4k,$$

where r_1, r_2 are the radii of the discs.

Calculate the minimum area of both discs together.

6. (a) Evaluate

(i) $\int_1^2 \frac{1 + x^2}{x^2}\,dx$ and $\int_0^1 \frac{x^2}{1 + x^2}\,dx$

(ii) $\int_{-2}^{0} (x + 1)e^{x(x + 2)}\,dx$

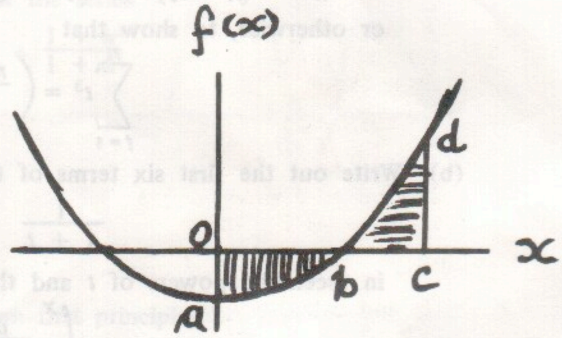
(iii) $\int_0^{\frac{\pi}{4}} [\tan x + \tan^3 x]\,dx.$

(b) The diagram shows a sketch of the function

$x \longrightarrow (3x - 4)(3x + 4).$

Find the coordinates of the point c such that

area of oab = area of bcd.



7. (a) Find the least integer n such that

$$\frac{n^2 + 3}{n^2 + 1} - 1 < 0{\cdot}01.$$

(b) Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!\ n^3}.$$

(c) $u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_r + \ldots\ldots\ldots$ is a convergent series of positive terms such that

$$1{\cdot}4 < \frac{u_r}{u_{r+1}} < 1{\cdot}5$$

for $r \geqslant 100$.

Show that

$$u_{101} < \tfrac{5}{7}\,u_{100}$$

and that

$$u_{101} + u_{102} + u_{103} + \ldots < 2{\cdot}5\,u_{100}.$$

8. (a) Define sample space (S), event (E) and singleton (E_i).

If $\#(S) = n$ and $\#(E) = r$, prove that

$$P(E) = \frac{r}{n},$$

where $P(E)$ means the probability of the event E.

Two dice are thrown and the numbers are noted. Write out the sample space for the experiment.

Find the probability that the sum thrown is

(i) equal to 8,

(ii) equal to at least 8.

(b) A serum has been developed and it has been found that 0·2% of the population suffer a bad reaction to it. In a random sample of 5000 people what is the probability that not more than 12 people will suffer a bad reaction ?

OR

8. (a) Complete the sentence:

"A vector $\vec{ab}$ is the set of all ....",

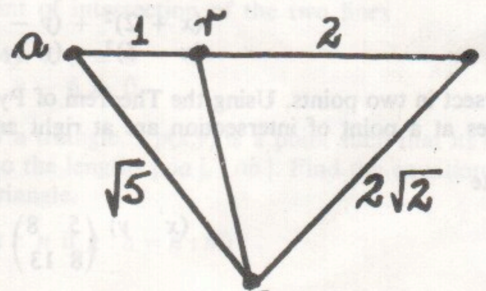
and explain how the vector $\vec{ab}$ can also be written as $\vec{k}$.

In the Δoab use the cosine formula (Tables p.9) to show that

$$\cos \angle aob = \frac{1}{\sqrt{10}}$$

and hence evaluate $\vec{a} \cdot \vec{b}$, taking o as the origin.

Write $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$ and calculate $|\vec{r}|$.

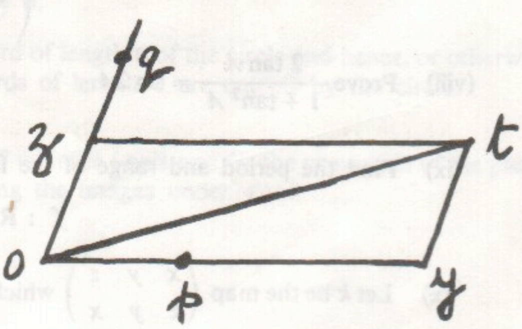


(b) Prove that for any three vectors $\vec{x}, \vec{y}, \vec{z}$

$$\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$$

where "." means scalar product.

$oytz$ is a parallelogram and p, q are any points on the lines oy and oz as in the diagram.



(i) Construct the locus of the vector $\vec{x}$ such that

$$\vec{x} \cdot \vec{y} = |oy| \cdot |op|$$

(ii) Show how to construct the point r on the line ot such that

$$|oy| \cdot |op| + |oz| \cdot |oq| = |ot| \cdot |or|.$$